Momentum Equation

The vector momentum equation can be written as

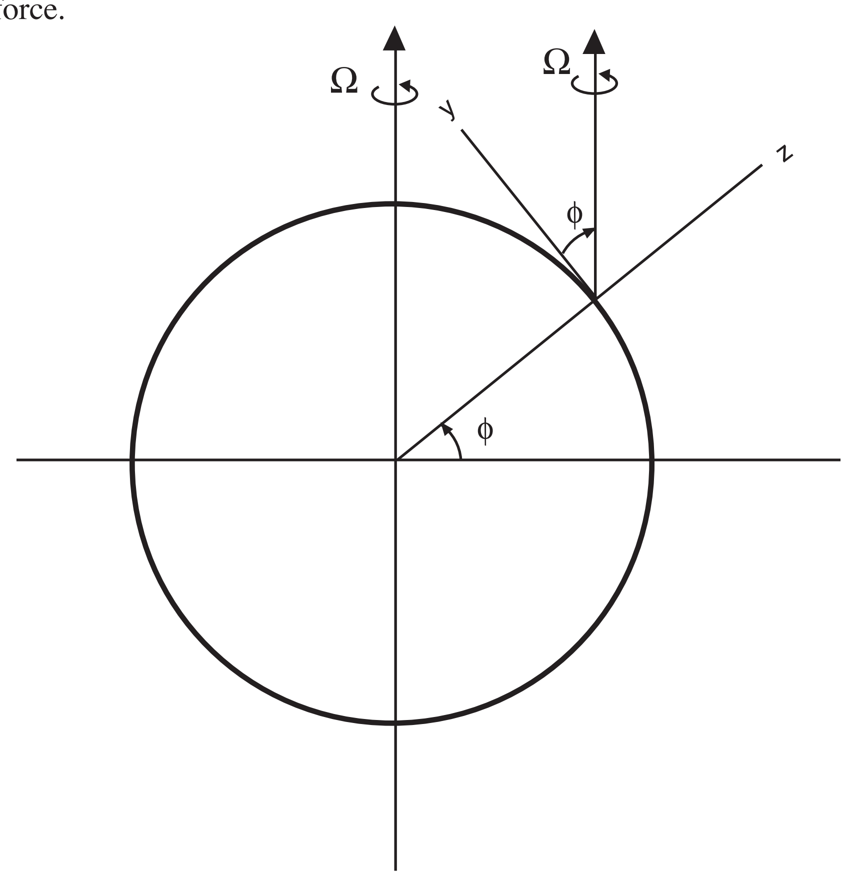
$$\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\nabla p - 2\vec{\Omega}\times\vec{V} + \vec{g} + \vec{F}_R$$

where

$\frac{d\vec{V}}{dt}$ =relative acceleration, $-\frac{1}{\rho}\nabla p$ =pressure gradient force, $2\vec{\Omega}\times\vec{V}$ =Coriolis force

$\vec{g}$ =gravity, and $\vec{F}_R$ =friction.

It is useful to break the above equation into component form. The only term that is a little tricky is the Coriolis force.



The earth's rotation has no component in the x-direction. Therefore, using the above figure

$$\vec{\Omega} = \Omega \cos\phi \vec{j} + \Omega \sin\phi \vec{k}$$

Then,

$$-2\vec{\Omega} \times \vec{V} = -2(\Omega \cos\phi \vec{j} + \Omega \sin\phi \vec{k}) \times (u\vec{i} + v\vec{j} + w\vec{k})$$

or after performing all of the cross products:

$$-2\vec{\Omega} \times \vec{V} = (2\Omega v \sin\phi - 2\Omega w \cos\phi)\vec{i} - 2\Omega u \sin\phi \vec{j} + 2\Omega u \cos\phi \vec{k}$$

Now it is possible to express the momentum equation in component form as,

$$\vec{i} \Rightarrow \frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v \sin\phi - 2\Omega w \cos\phi + F_{RX}$$

$$\vec{j} \Rightarrow \frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u \sin\phi + F_{RY}$$

$$\vec{k} \Rightarrow \frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega u \cos\phi + F_{RZ}$$

The above equation is valid for ANY fluid within a rotating frame of reference.

LARGE-SCALE QUASI-HORIZONTAL FLOW

At the present time, we are only interested in large-scale quasi-horizontal flow (i.e., the earth's atmosphere) which will simplify the above equation. We perform what is referred to as a "scale analysis" of the terms. This is nothing more than a fancy way of saying that there are certain terms that we can ignore. We just need to figure out what those terms are.

For large-scale quasi-horizontal flow, we can say the following:

$U \sim 10$ m/s	typical horizontal wind speed in the atmosphere
$W \sim 10^{-2}$ m/s=1 cm/s	typical vertical wind speed
$L \sim 10^{6}$ m=1000 km	length scale (approximates an extratropical cyclone)
$H \sim 10^{4}$ m= 10 km	depth of the troposphere
$T = L/U \sim 10^{5}$ s~ 1 day	time scale
$2\Omega \sim 10^{-4}$ s^{-1}	Coriolis force.

Let's first examine the vertical component of the equation.

$$\cancel{\frac{dw}{dt}} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \cancel{+ 2\Omega u \cos\phi} \cancel{+ F_{RZ}}$$

UW/L			2ΩU	
10^{-7} m/s^2	10m/s^2	10 m/s^2	10^{-3} m/s^2	10^{-15} m/s^2

$$\frac{\partial p}{\partial z} = -\rho g$$

This tells us that for large scale flow, the atmosphere is in **hydrostatic balance**. If we perform the same scale analysis for the horizontal equation of motion, you will find that we should drop the friction term and $2\Omega w\cos\phi$.

This leaves us with the following horizontal equations:

$$\vec{i} \Rightarrow \frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv$$

$$\vec{j} \Rightarrow \frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu$$

$$where \quad f = 2\Omega \sin\phi$$

f is called the **Coriolis parameter** and is >0 in the northern hemisphere and <0 in the southern hemisphere.

If we try to combine the component equations into a horizontal vector equation, we have

$$\frac{du}{dt}\vec{i} + \frac{dv}{dt}\vec{j} = -\frac{1}{\rho}\frac{\partial p}{\partial x}\vec{i} - \frac{1}{\rho}\frac{\partial p}{\partial y}\vec{j} - fu\vec{j} + fv\vec{i}$$

or

$$\frac{d\vec{V}_H}{dt} = -\frac{1}{\rho}\nabla_H p - f\vec{k} \times (u\vec{i} + v\vec{j})$$

or

$$\boxed{\frac{d\vec{V}_H}{dt} = -\frac{1}{\rho}\nabla_H p - f\vec{k} \times \vec{V}_H}$$

Once again, it is very convenient to use natural coordinates to help us understand the kind of flows that are in the atmosphere. If we write the above equation in natural coordinates rather than Cartesian coordinates, we will find:

$$\vec{t} \Rightarrow \frac{dV_H}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial s}$$

$$\vec{n} \Rightarrow \frac{V_H^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial n} - fV_H$$

where,

$\frac{dV_H}{dt}$ = tangential acceleration $\qquad -\frac{1}{\rho}\frac{\partial p}{\partial s}$ = tangential pressure gradient force

$\frac{V_H^2}{R}$ = centripetal acceleration $\qquad -\frac{1}{\rho}\frac{\partial p}{\partial n}$ = normal pressure gradient force

$-fV_H$ = Corolis force.

GEOSTROPHIC WIND

Often times the real winds often appear to be non-accelerating. Let's see how the equations of motion in natural coordinates simplify when we make this assumption (i.e., the tangential $\frac{dV_H}{dt}$ and the normal acceleration $\frac{V_H^2}{R}$ are set to zero). We are left with

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial n} - fV_g$$

or if we rewrite this equation,

$$V_g = -\frac{1}{\rho f}\frac{\partial p}{\partial n}$$

This is known as the **geostrophic wind equation**. We can write the geostrophic equation in vector form by the following method,

$$\frac{d\vec{V}_H}{dt} = 0 = -\frac{1}{\rho}\nabla_H p - f\vec{k} \times \vec{V}_g$$

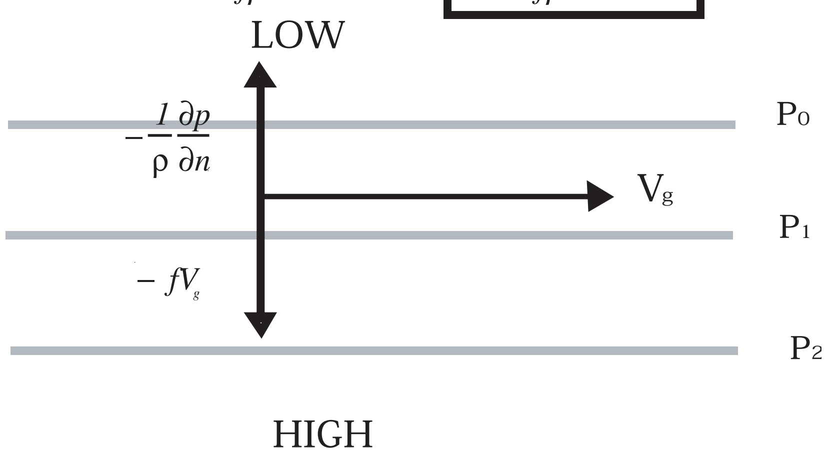
$$\vec{k} \times \vec{V}_g = -\frac{1}{f\rho}\nabla_H p$$

.

We then use a little vector trick:

$$\vec{k} \times (\vec{k} \times \vec{V}_g) = -\frac{1}{f\rho}\vec{k} \times \nabla_H p$$

$$-\vec{V}_g = -\frac{1}{f\rho}\vec{k} \times \nabla_H p \quad or \quad \boxed{\vec{V}_g = \frac{1}{f\rho}\vec{k} \times \nabla_H p}$$



The wind at 500 mb will be approximately geostrophic since friction is negligible. You probably heard the saying that "with your back to the wind in the northern hemisphere, low pressure will be on your left."

There are a couple of simple rules about geostrophy if you examine the equation:

1. The stronger the pressure gradient force (i.e., the more closely spaced the isobars), the stronger the geostrophic wind.

2. For the same pressure gradient force, the geostrophic wind will be weaker at higher latitudes.

GRADIENT WIND

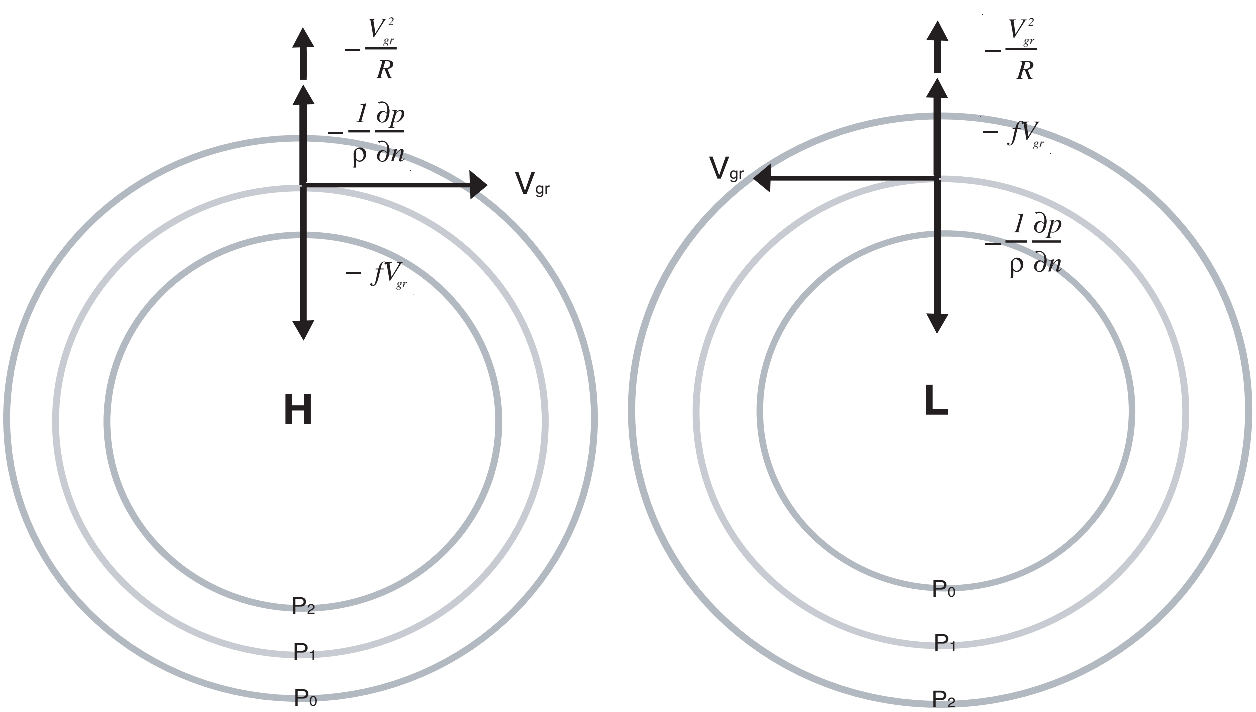
For the next type of wind, we will assume there is no tangential acceleration ($\frac{dV_H}{dt}$=0), but we will allow for the flow to be curved ($\frac{V_H^2}{R} \neq 0$). In other words, we do allow for normal or centripetal accelerations. Our equation in natural coordinates reduces to

$$\frac{V_{gr}^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial n} - fV_{gr}$$

which is sometimes referred to as the **gradient wind equation**. This is a quadratic equation for V_{gr} and has the following solution:

$$V_{gr} = -\frac{fR}{2} \pm \sqrt{(\frac{fR}{2})^2 - \frac{R}{\rho}\frac{\partial p}{\partial n}}.$$

We see that there are 2 solutions that satisfy the equation, but only one of them makes any sense.



The gradient wind is used to explain the basic flows around High and Low pressure centers.

If you study the flow patterns carefully, you will note a problem with the balance of forces around a High when the wind becomes too strong. When winds become intense, it is impossible for the Coriolis force to balance the centripetal acceleration. The reason is simple: one is proportional to the velocity squared. In other words, if the wind becomes too strong, the forces around a High cannot maintain balance. We see this in actual Highs on surface charts. The pressure gradient force near the center of the High is almost always quite weak (i.e., the wind is weak in the center of the High). No such restriction exists for Low pressures.

CYCLOSTROPHIC WINDS

When the wind becomes intense around a Low pressure center (e.g., a tornado), the Coriolis force no longer is important. The equation then reduces to:

$$\frac{V_{cy}^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial n}.$$

This is called the **cyclostrophic equation** and is approximately valid for a tornado. So, intense circulations are simply a balance between the pressure gradient force and the centripetal acceleration. Since the Coriolis force is no longer important, the wind can circulate clockwise or counterclockwise around the Low (i.e., is is possible to have an anticyclonic tornado).

THERMAL WIND

In the atmosphere, it is often very important to understand how the wind changes with height, often referred to as "wind shear." Mathematically, this would mean we would have to solve for $\frac{\partial \vec{V}_H}{\partial z}$. This is not an easy equation to evaluate. Since atmospheric winds are "quasi"-geostrophic, we make our life simpler by evaluating $\frac{\partial \vec{V}_g}{\partial z}$ instead. Be careful. By making this approximation, we assume that the wind is not accelerating.

$$\vec{V}_g = \frac{1}{f\rho}\vec{k} \times \nabla_H \; p$$

$$\frac{\partial \vec{V}_g}{\partial z} = -\frac{1}{\rho^2 f}\frac{\partial \rho}{\partial z}\vec{k} \times \nabla_H p + \frac{1}{\rho f}\vec{k} \times \nabla_H (\frac{\partial p}{\partial z})$$

use the hydrostatic equation

$$\frac{\partial \vec{V}_g}{\partial z} = (-\frac{1}{\rho}\frac{\partial \rho}{\partial z})(\frac{1}{\rho f}\vec{k} \times \nabla_H \; p) + \frac{1}{\rho f}\vec{k} \times \nabla_H (-\rho g)$$

the term in the second parentheses is the geostrophic wind

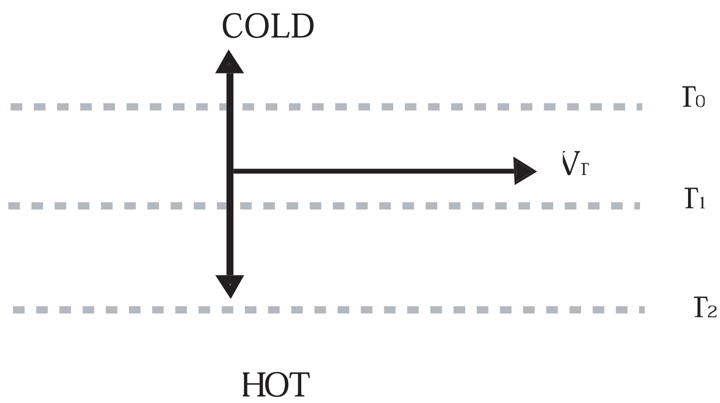
$$\frac{\partial \vec{V}_g}{\partial z} = (-\frac{1}{\rho}\frac{\partial \rho}{\partial z})\vec{V}_g - \frac{g}{\rho f}\vec{k} \times \nabla_H (\rho)$$

Finally, it is better to have this equation written in terms of temperature rather than density. This can be done by using the equation of state ($p = \rho RT$) so that,

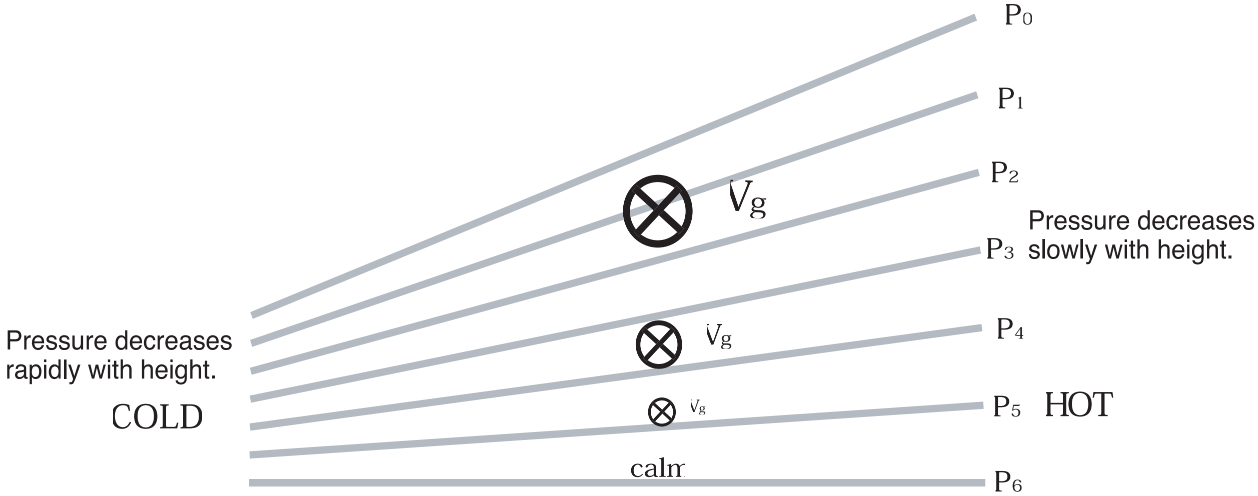
$$\vec{V}_T = \frac{\partial \vec{V}_g}{\partial z} = \frac{1}{T}\frac{\partial T}{\partial z}\vec{V}_g + \frac{g}{fT}\vec{k} \times \nabla_H (T) \approx \frac{g}{fT}\vec{k} \times \nabla_H (T)$$

This is the **thermal wind equation** where we have shown that it is the second term that dominates.

This equation looks remarkably similar to the geostrophic wind equation except we have replaced isobars with isotherms. This tells us that the change of geostrophic wind with height is related to the temperature field. The following diagrams attempt to explain why this is the case.



Horizontal view



Vertical view

<u>EQUATION OF CONTINUITY</u>

The equation of continuity is an equation that satisfies mass conservation (i.e., mass cannot be created or destroyed). The derivation is not difficult (see Holton's book), but we will just provide the equation at this point.

$$\frac{\partial \rho}{\partial t} + \nabla \bullet \rho \vec{V} = 0$$

The above form of the continuity equation tells us that the density at a fixed point changes due to the divergence ($\nabla \bullet$) of the momentum per unit volume ($\rho \vec{V}$). Another form of this equation can be found by performing the following operations,

$$\frac{\partial \rho}{\partial t} + \nabla \bullet \rho \vec{V} = \frac{\partial \rho}{\partial t} + \vec{V} \bullet \nabla \rho + \rho \nabla \bullet \vec{V} = 0$$

$$or$$

$$\frac{d\rho}{dt} + \rho \nabla \bullet \vec{V} = 0 \quad or$$

$$\frac{1}{\rho}\frac{d\rho}{dt} + \nabla \bullet \vec{V} = 0$$

This equation tells us that the fractional rate of change of the density is related to the divergence of the velocity. Another way we can view this equation is to realize that the density is related to the mass divided by the volume.

$$\rho = \frac{M}{Vol} \quad and \quad ln\,\rho = ln\,M - ln\,Vol$$

$$\frac{1}{\rho}\frac{d\rho}{dt} + \nabla \bullet \vec{V} = \frac{d(ln\,\rho)}{dt} + \nabla \bullet \vec{V} = \cancel{\frac{1}{M}\frac{dM}{dt}} - \frac{1}{Vol}\frac{dVol}{dt} + \nabla \bullet \vec{V} = 0$$

$$\nabla \bullet \vec{V} = \frac{1}{Vol}\frac{dVol}{dt}$$

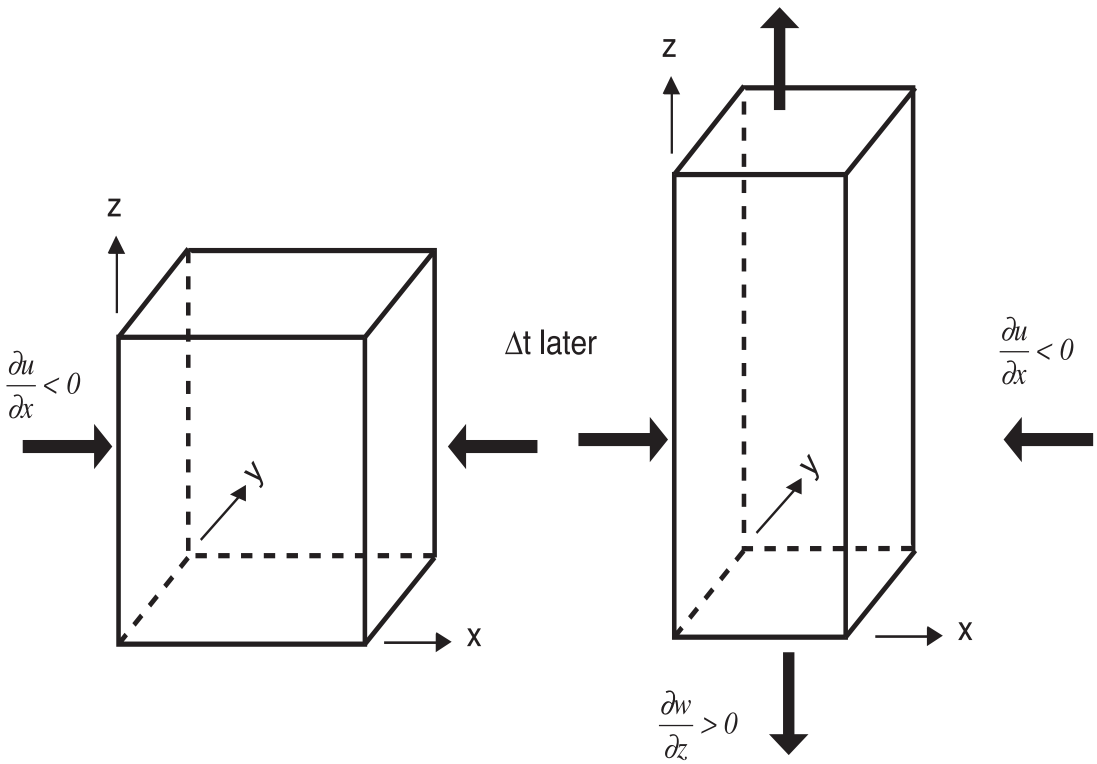
This equation tells us that the velocity divergence is related to the fractional rate of change of a parcel's volume.

It is often useful to talk about a fluid that is *incompressible*. This is stated mathematically by saying $\frac{d\rho}{dt} = 0$ (the density following a parcel does not change). The atmosphere is approximately incompressible except when sound waves are generated or thunder associated with lightning occurs. These two phenomena "compress" the air.

If the atmosphere is incompressible, then the continuity equation simplifies to

$$\nabla \bullet \vec{V} = \frac{\partial u}{\partial} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{1}{Vol}\frac{dVol}{dt} = 0 .$$

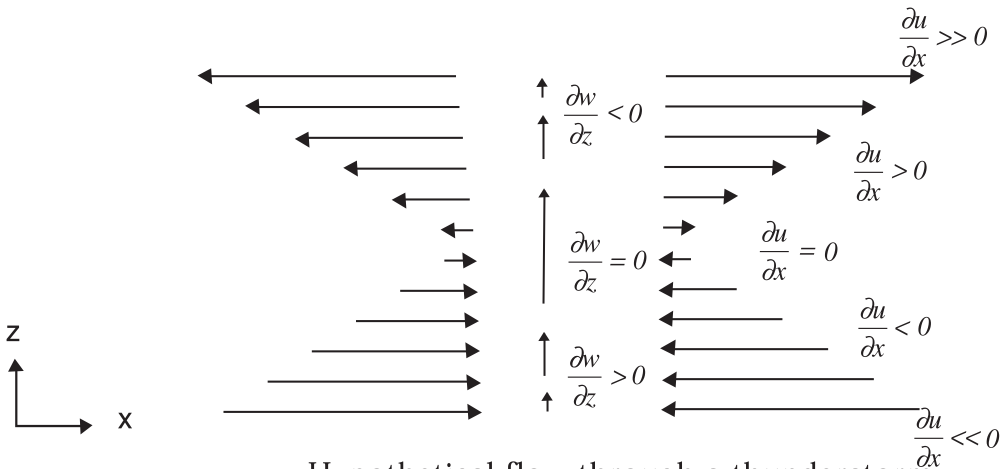
We now see that velocity divergence satisfies the continuity equation when the flow is incompressible. We also see that the volume of a parcel cannot change with time if the flow is incompressible. An example is shown below.



The volume of the box cannot change if the parcel is incompressible. Horizontal convergence in the x-direction at the initial time means that the box must stretch in the z-direction.

In the above example, we have assumed that v=0.

Let us also examine a two-dimensional case of flow through a thunderstorm assuming that it is incompressible.



Hypothetical flow through a thunderstorm

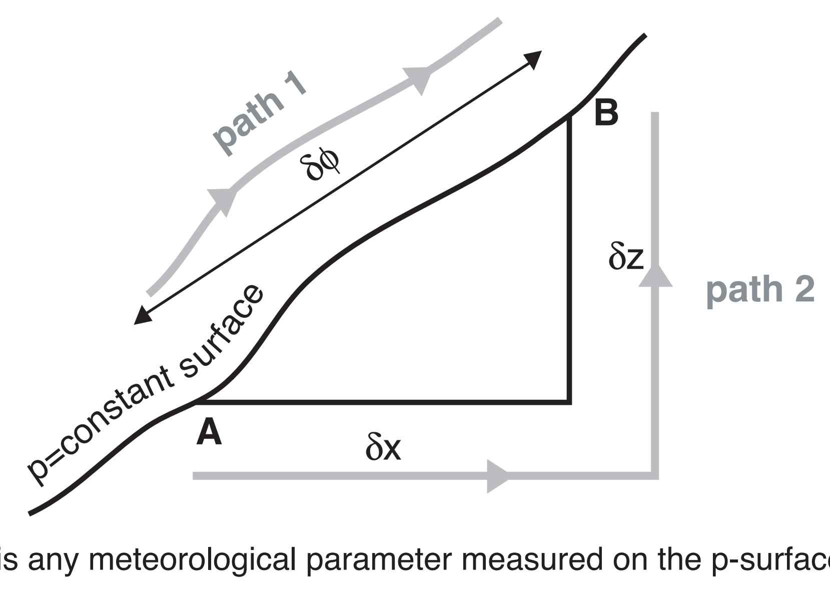
$$\nabla \bullet \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} = -\frac{\partial w}{\partial z}$$

Note that w is a max where $\frac{\partial w}{\partial z} = 0$ as would be expected from basic calculus.

PRESSURE AS THE VERTICAL COORDINATE

In the atmosphere it is often useful to change the vertical coordinate from height (or z) to pressure (or p). For example, we often talk about the 500-mb surface which is really using pressure as a vertical coordinate.



ϕ is any meteorological parameter measured on the p-surface.

We can see that there are two paths we can take on the above diagram to travel from point A to point B.

$$path\ 1 \Rightarrow (\delta\phi)_{p=const}$$

$$path\ 2 \Rightarrow \frac{\partial\phi}{\partial x})_{y,z,t}\delta x + \frac{\partial\phi}{\partial z})_{x,y,t}\delta z$$

since they have to provide the same answer –

$$(\delta\phi)_{p=const} = \frac{\partial\phi}{\partial x})_{y,z,t}\delta x + \frac{\partial\phi}{\partial z})_{x,y,t}\delta z \quad or \quad dividing \quad by \quad \delta x$$

$$(\frac{\partial\phi}{\partial x})_{p=const} = \frac{\partial\phi}{\partial x})_{y,z,t} + \frac{\partial\phi}{\partial z})_{x,y,t}\frac{\partial z}{\partial x}$$

Please do not confuse $(\frac{\partial\phi}{\partial x})_{p=const}$ with $\frac{\partial\phi}{\partial x})_{y,z,t}$. The former is evaluated on a *p*-surface while the latter is evaluated on the *x*-axis.

Similarly, we would find the following equation for the *y*-direction

$$(\frac{\partial\phi}{\partial y})_{p=const} = \frac{\partial\phi}{\partial y})_{x,z,t} + \frac{\partial\phi}{\partial z})_{x,y,t}\frac{\partial z}{\partial y} \quad .$$

Let's try to make a vector equation by adding these two equations.

$$(\frac{\partial\phi}{\partial x})_{p=const}\vec{i} + (\frac{\partial\phi}{\partial y})_{p=const}\vec{j} = \frac{\partial\phi}{\partial x})_{y,z,t}\vec{i} + \frac{\partial\phi}{\partial y})_{x,z,t}\vec{j} + \frac{\partial\phi}{\partial z})_{x,y,t}\frac{\partial z}{\partial x}\vec{i} + \frac{\partial\phi}{\partial z})_{x,y,t}\frac{\partial z}{\partial y}\vec{j}$$

or

$$\nabla_P\phi = \nabla_H\phi + \frac{\partial\phi}{\partial z}\nabla_P z$$

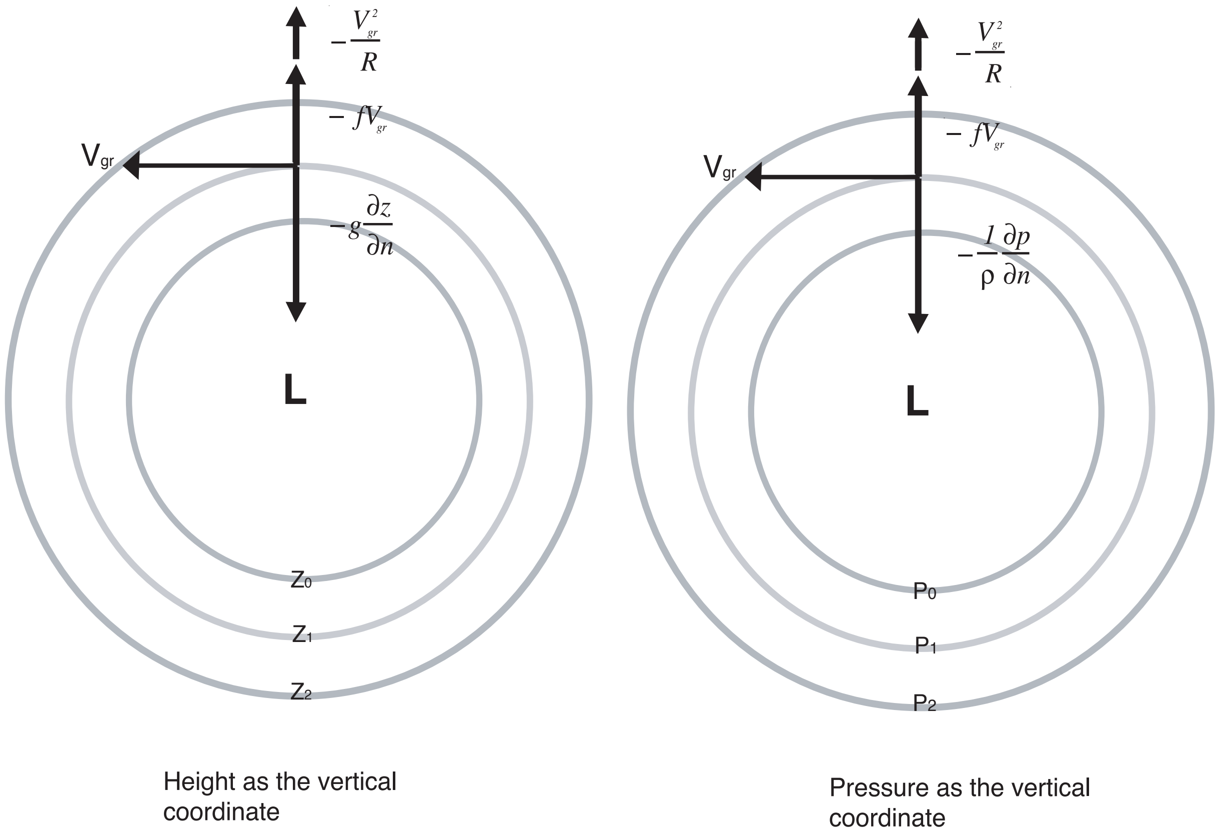
Now assume that the meteorological parameter we want to measure is pressure (i.e.,ϕ=*p*). Then,

$$\nabla_P\, p = 0 = \nabla_H\, p + \frac{\partial p}{\partial z}\nabla_P\, z$$

where we have used the fact that the change of *p* on a *p*-surface must be zero. Rearranging terms and using the hydrostatic equation, we have

$$\boxed{-\frac{1}{\rho}\nabla_H\, p = -g\nabla_P\, z} \quad .$$

This important equation tells us that the pressure gradient force in a height coordinate system is equivalent to the gradient of height on a pressure coordinate system.



An advantage of using a pressure coordinate system is that the density is no longer included. This relieves us of the need to measure a difficult quantity.

We next have to talk about the vertical velocity in a pressure coordinate system. In a height coordinate system,

$$w = \frac{dz}{dt}$$

Accordingly, the vertical velocity in a pressure coordinate system is $\frac{dp}{dt}$. We call this omega (ω) and the equation is,

$$\omega = \frac{dp}{dt}$$

Using the definition of the total derivative,

$$\omega = \frac{dp}{dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} \approx w\frac{\partial p}{\partial z}.$$

We have used the fact that it is the last term that dominates. This means that

$$\omega \approx -\rho g w \quad .$$

This tell us that the *w* and ω are opposite in sign, which makes sense since *p* decreases upward. The normal units for ω are μb/sec. Just for you information, 1 μb/sec is equivalent to 86 mb/24hr.

It should also be noted that it is possible for w=0 but $\omega \neq 0$ since the *p*-surfaces could be moving.

Let's now see how some of our equations change in the pressure coordinate system.

	HEIGHT	PRESSURE
geostrophic wind	$\vec{V}_g = \frac{1}{f\rho}\vec{k} \times \nabla_H p$	$\vec{V}_g = \frac{g}{f}\vec{k} \times \nabla_H z$
gradient wind	$\frac{V_{gr}^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial n} - fV_{gr}$	$\frac{V_{gr}^2}{R} = -g\frac{\partial z}{\partial n} - fV_{gr}$
thermal wind	$\vec{V}_T = \frac{\partial \vec{V}_g}{\partial z} = \frac{1}{T}\frac{\partial T}{\partial z}\vec{V}_g + \frac{g}{fT}\vec{k} \times \nabla_H (T)$	$\vec{V}_T = -\frac{\partial \vec{V}_g}{\partial p} = \frac{R}{fp}\vec{k} \times \nabla_P (T)$
continuity equation	$\frac{1}{\rho}\frac{d\rho}{dt} + \nabla \bullet \vec{V} = 0$	$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$

It is only the geostrophic wind in *p*-coordinates that is important to remember at this point. I have provided the others for completeness even though I didn't derive them.