Rethinking mode splitting, splitting in general, Boussinesq, non-Boussinesq, seawater EOS, and how it all comes together

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A *convoluted story* about how making code work comes long before understanding of why it works, and understanding why it works leads to rediscovery of old knowledge. Essentially a unified approach to mathematical splitting of stiff operators, not just for barotropic-baroclinic modes, but overall throughout the oceanic solver. A perturbational analysis helps to derive two- or three-way splits involving several components of the model. Barotropic mode, compressible EOS, implicit bottom drag can peacefully co-exist without overwriting computational efforts of each other.

Honolulu, HI, April 2010

Motivation

- Finalize the design of Boussinesq code: stiffened EOS, clean, self-consistent Boussinesq approximation
- Justification?
- *Reviewer*: Is it about EOS? Then what EOS has to do with mode splitting? Other models go non-Boussinesq with ease. Why don't you?

The *sole purpose* of Boussinesq approximation in a numerical model is to facilitate *splittings*.

- What is splitting?

- What is Boussinesq approximation?

This presentation is organized as follows:

- Splitting overview
- Splitting example: Implicit bottom drag
- Boussinesq approximation and seawater compressibility
- EOS in Boussinesq ROMS
- Splitting again: Issues with barotropic-baroclinic mode splitting for non-Boussinesq model with compressible EOS
- Looking back to Boussinesq model

Splitting, a.k.a. fractional stepping

- Classical update-operator splitting

$$\partial_t \mathbf{u} = \mathcal{R}(\mathbf{u}) \qquad \text{where} \qquad \mathcal{R}(\mathbf{u}) = \mathcal{R}_1(\mathbf{u}) + \mathcal{R}_2(\mathbf{u}) \qquad \text{but straightforward}$$

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t \cdot \mathcal{R}\left(\mathbf{u}^{n:n+1}\right)$$

is not practical because of complexity (implicitness), so instead

$$\mathbf{u}' = \mathbf{u}^n + \Delta t \cdot \mathcal{R}_1(\mathbf{u}^{n:\prime}) \qquad \text{followed by} \qquad \mathbf{u}^{n+1} = \mathbf{u}' + \Delta t \cdot \mathcal{R}_2(\mathbf{u}^{\prime:n+1})$$

$$\mathbf{u}^{n+1} = [1 + \Delta t \cdot \mathcal{R}_2(.)] \cdot [1 + \Delta t \cdot \mathcal{R}_1(.)]\, \mathbf{u}^n$$

$$[1 + \Delta t \cdot \mathcal{R}_2(.)] \cdot [1 + \Delta t \cdot \mathcal{R}_1(.)] \neq [1 + \Delta t \cdot \mathcal{R}_1(.)] \cdot [1 + \Delta t \cdot \mathcal{R}_2(.)]$$

resulting in $\mathcal{O}(\Delta t)$ operator splitting error... ...**unless carefully designed**

Examples? **everywhere**:

- Barotropic-baroclinic mode splitting, either split-explicit or implicit free surface
- Directional splitting: computationally efficient way to introduce cross terms to stabilise forward-in-time upstread-biased advection
- Forward-backward time stepping $\Rightarrow$ gain in accuracy
- Semi-implicit viscosity/diffusion
- Deferred (or lagged) explicit lateral viscous terms

- Biological models in ROMS use *splitting by physical processes*, partially implicit $\Rightarrow$ stability + non-negativity of concentrations

Difficulties:

- Both R_1 and R_2 are stiff (implicit, internally balanced large terms). Especially inaccurate in near cancellation $R_1 \approx -R_2$ situation (balance).
- long-standing dilemma of no-slip boundaries + pressure-Poisson projection method for incompressible flows

Splitting is present from day one in ocean modeling...

Marchuk, G. I., 1964: Theoretical model for weather forecasting. *Dokl. Akad. SSSR*, **155**, 1062-1065.

Crowley, W. P., 1968: A global numerical ocean model. *J. Comput. Phys*, **3**, 111-147.

Bryan, K., and M. Cox, 1969: A numerical method for the study of the circulation of the world ocean. *J. Comput. Phys*, **4**, 347-376.

Simons, T. J., 1974: Verification of numerical models of lake Ontario. 1. Circulation in spring and early summer. *J. Phys. Oceanogr.*, **4**, 507-523.

Rangarao V. Madala, R. V. and S. A. Piacsek, 1977: A semi-implicit numerical model for baroclinic oceans *J. Comput. Phys*, **23**, 167-178.

Berntsen, H., Z. Kowalik, S. Sælid, and K. Sørli, 1981: Efficient numerical simulation of ocean hydrodynamics by a splitting procedure. *Modeling, Identification and Control*, **2**, No. 4, 181-199.

Blumberg, A. F. and G. L. Mellor, 1987: A description of a three-dimensional coastal ocean circulation model. In *Three-Dimensional Coastal Ocean Models*, ed. N. Heaps (Pub. AGU), 1-16.

Bleck, R. and L. T. Smith, 1990: A wind-driven isopycnic coordinate model of the north and equatorial Atlantic Ocean: 1. Model development and supporting experiments. *J. Geophys. Res.*, **95C** 3273-3285.

Killworth, P. D., D. Stainforth, D. J. Webb and S. M. Paterson, 1991: The development of a free-surface Bryan-Cox-Semtner ocean model. *J. Phys. Oceanogr.*, **21**, 1333-1348.

Casulli, V. and R. T. Cheng, 1992: Semi-implicit finite-difference methods for 3-dimensional shallow-water flow. *Int. J. Numer. Meth. Fluids.*, **15**, 629-648.

Marshall, J., A. Adcroft, C. Hill, L. Perelman, and C. Heisey, 1997: A finite-volume, incompressible Navier Stokes model for studies of the ocean on parallel computers. *J. Geophys. Res.*, **102**, 5753-5766.

Implicit bottom drag dilemma

Implicit no-slip boundary condition is essentially a **statement of balance** bottom stress $-\Delta t \cdot r_D \cdot u_{k=1}^{n+1}$ instantaneously adjusts itself to keep $u_{z\to\text{bottom}} = 0$. Stress computation must be included into implicit solver for vertical viscosity terms, however this interferes with Barotropic Mode (BM) splitting:

- Bottom drag can be computed only from full 3D velocity, but not from the vertically averaged velocities alone.
- Barotropic Mode must know the bottom drag term **in advance** as a part of 3D→2D forcing for consistency of splitting. This places computing vertical viscosity before BM, however, later when BM corrects the vertical mean of 3D velocities, it *destroys* the consistency of (no-slip like) bottom boundary condition.
- If BM receives bottom drag based on the most recent state of 3D velocity **before** BM, but the implicit vertical viscosity terms along with (the final) bottom drag are computed **after** BM is complete (hence accurately respecting the bottom boundary condition), this changes the state of vertical integrals of 3D velocities, interfering with BM in keeping the vertically integrated velocities in nearly non-divergent state.
- Current ROMS practice is to split bottom drag term from the rest of vertical viscosity computation. This limits the time step (or r_D itself) by the explicit stability constraint.

Ekman layer in shallow water: $h = 10m$, $u_* = 6\times10^{-2}m/s$ ($\approx 5m/s$ wind), $f = 10^{-4}$, $A_v = 2\times10^{-3}m^2/s$, non-slip at $z = -h$

Top: Explicit, CFL-limited, bottom drag **before** Barotropic Mode (BM) for **both** r.h.s. 3D and for BM forcing ($\Rightarrow$ no splitting error); implicit step for vertical viscosity **after** with bottom drag excluded ($\Rightarrow$ undisturbed coupling of 2D and 3D); need $r_D < \Delta z_{\text{bottom}}/\Delta t_{\text{3D}}$ for stability

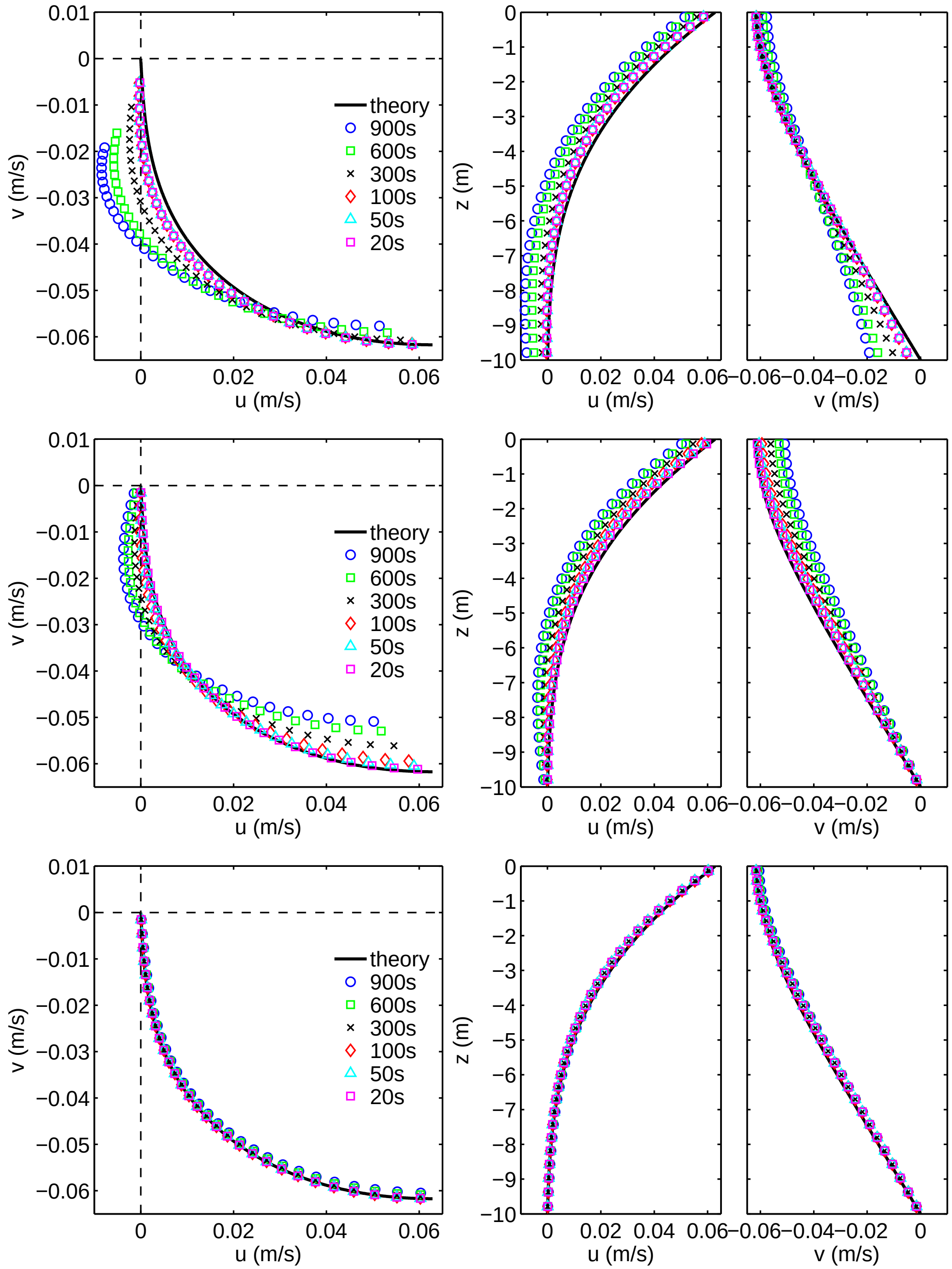
Middle: Unlimited drag **before** BM applies for BM forcing **only**; implicit vertical viscosity **after** with drag included into implicit solver (i.e., the drag is recomputed relative to what BM got before $\Rightarrow$ splitting error)

Bottom: Bottom drag is computed as a part of implicit vertical viscosity step **before** and for **both** 3D and BM forcing

In all cases BM has bottom drag term which captures its tendency in fast time

$$\partial_t\overline{\mathbf{U}} = \ldots \underbrace{\Big[\underbrace{-r_D\cdot\mathbf{u}_{\text{bottom}}}_{\text{drag from 3D mode}} + r_D\cdot\overline{\mathbf{u}}^{m=0}\Big]}_{\text{3D}\rightarrow\text{BM forcing}} - r_D\cdot\overline{\mathbf{u}}$$

so when $\mathbf{u}_{\text{bottom}}$ is updated/corrected by BM, so does the $-r_D\cdot\mathbf{u}_{\text{bottom}}$ term computed from it; above $\overline{\mathbf{U}} = (h+\zeta)\overline{\mathbf{u}}$



- This is a **stationary problem**, but the results depend on Δt**(!)**
- 3-way balance: (1) Coriolis vs. vertical viscosity (Ekman layer); (2) wind stress vs. bottom drag; (3) barotropic mode keeps flow 2D non-divergent
- A one-dimensional – vertical column with co-located u, v – can be trivially solved as tri-diagonal system. In ROMS this is prevented by C-grid staggering of u and v, and barotropic splitting
- sub-step mutually perturb each other. This is countered by including cross-tendency terms
- Requires substantial redesign of ROMS kernel
- somewhat encourages **anti-modular** code design
- Possible only in corrector-coupled and Generalized FB variants of ROMS kernels
- Incompatible (or at least hard to implement) in Rutgers kernel because of forward extrapolation of r.h.s. terms for 3D momenta (AB3 stepping) and extrapolation of 3D→BM forcing terms which is not compatible with having stiff terms there
- Incompatible with predictor-coupled kernel (currently used by AGRIF), because of extrapolation of 3D→BM forcing, and because overall having BM too early the computing sequence (implicit vertical viscosity step is done only after predictor step for tracers which is after BM)

Do we need implicit bottom drag?

- Unlike "toy problem", turbulent drag coefficients are space- and velocity- dependent and not known *a priori*. The **discretized** model needs

$$\Delta z_1 \cdot \frac{u_1^{n+1} - u_1^n}{\Delta t} = A_{3/2} \cdot \frac{u_2^{n+1} - u_1^{n+1}}{\Delta z_{3/2}} - r_D \cdot u_1^{n?} \qquad r_D = ?$$

where $u_1 \equiv u_{k=1}$ is understood in finite-volume sense $u_1 = \frac{1}{\Delta z_1} \int_{\text{bottom}}^{\text{bottom}+\Delta z_1} u\left(z'\right)\, \mathrm{d}z'$

- from physics: $\text{STRESS} = F(u)$, $F = ?$

- duality of u_*: it controls **both** bottom stress and vertical viscosity profile

$$\text{STRESS} = u_*^2\,, \qquad \text{and} \qquad A = A(z) = \kappa u_* \cdot (z_0 + z) \qquad z \to 0$$

roughness length z_0 = statistically averaged scale of unresolved of topography

- constant-stress boundary layer $A(z) \cdot \partial_z u = \text{STRESS} = const = u_*^2$

$$\kappa u_* \left(z_0 + z\right) \partial_z u = u_*^2 \qquad \text{hence} \qquad u(z) = \frac{u_*}{\kappa} \ln\left(1 + \frac{z}{z_0}\right)$$

$$u_1 = \frac{u_*}{\kappa}\left[\left(\frac{z_0}{\Delta z_1} + 1\right) \ln\left(1 + \frac{\Delta z_1}{z_0}\right) - 1\right] \qquad \text{hence} \qquad u_* = \kappa \cdot u_1/[...]$$

$$-r_D \cdot u_1 = -\kappa^2 \left|u_1\right| \cdot \left[\left(\frac{z_0}{\Delta z_1} + 1\right) \ln\left(1 + \frac{\Delta z_1}{z_0}\right) - 1\right]^{-2} \cdot u_1$$

$$r_D = \kappa^2 |u_1| \Bigg/ \left[\left(\frac{z_0}{\Delta z_1} + 1 \right) \ln \left(1 + \frac{\Delta z_1}{z_0} \right) - 1 \right]^2$$

well-resolved asymptotic limit for $\Delta z_1 / z_0 \ll 1$ is $\quad r_D \sim 4\kappa^2 |u_1| \cdot \dfrac{z_0^2}{\Delta z_1^2}$

however in this case $\quad u(z) = \dfrac{u_*}{\kappa} \ln\left(1 + \dfrac{z}{z_0}\right) \sim \dfrac{u_*}{\kappa} \cdot \dfrac{z}{z_0} \quad$ hence $\quad u_1 = \dfrac{u_*}{\kappa} \cdot \dfrac{\Delta z_1}{2 z_0}$

resulting $\quad r_D \sim \kappa^2 u_* \cdot \dfrac{2 z_0}{\Delta z_1} = \dfrac{A_{\text{bottom}}}{\Delta z_1 / 2} \quad$ in line with no-slip with laminar viscosity

unresolved $\Delta z_1 / z_0 \gg 1$ limit $\quad r_D \sim \kappa^2 |u_1| \Big/ \ln^2 \left(\dfrac{\Delta z_1}{z_0} \right) \quad$ known as "log-layer"

- overall there is nothing unexpected
- smooth transition between resolved and unresolved
- avoids introduction of *ad hoc* "reference height" z_a, e.g., Soulsby (1995) formula $\quad \mathrm{STRESS} = [\kappa / \ln(\mathrm{z_a}/\mathrm{z_0})]^2 \cdot \mathrm{u}^2\big|_{\mathrm{z=z_a}} \quad$ where $u\big|_{z=z_a}$ is hard (or impossible) to estimate from discrete variables
- in practice this differs by a factor of 2 from published formulas, e.g., Blaas (2007), with $z_a = \Delta z_1 / 2$, due to finite-volume vs. finite-difference interpretation of discrete model variables
- near-bottom vertical grid-box height Δz_1 is an inherent control parameter of r_D, making it impossible to specify "physical" quadratic drag coefficient, $r_D = C_{\mathrm{D}} \cdot |u|$

How large is $\dfrac{\Delta t \cdot r_D}{\Delta z_1}$?

$$\frac{\Delta t \cdot r_D}{\Delta z_1} = \underbrace{\frac{\Delta t \cdot |u_1|}{\Delta x}}_{\text{advective Courant number}} \cdot \kappa^2 \cdot \underbrace{\frac{\Delta x}{\Delta z_1} \Bigg/ \left[\left(\frac{z_0}{\Delta z_1} + 1 \right) \ln \left(1 + \frac{\Delta z_1}{z_0} \right) - 1 \right]^2}_{\text{purely geometric criterion}}$$

in unresolved case $\quad \dfrac{\Delta x}{\Delta z_1} \cdot \left[\kappa \Big/ \ln \left(\dfrac{\Delta z_1}{z_0} \right) \right]^2$

Typical high-resolution ROMS practice $h_{\text{min}} \sim 25m$, $N = 30...50$, hence $\Delta z \sim 1m$, $\Delta x = 1km$, and $z_0 = 0.01m$, $\kappa = 0.4$ estimates the above as 7.5.

- $\sim 50...100$ **in Bering Sea in our $\Delta x = 12.5$km Pacific simulation, even more in a coarser 1/5-degree**

It is mitigated by the bottom-most velocity Courant number ~ 0.1 but, still exceeds the limit of what explicit treatment can handle

- sigma-models are the most affected, but they are the ones which are mostly used when bottom drag matters
- vertical grid refinement toward the bottom makes this condition stiffer

Boussinesq approximation: Overview

Boussinesq, J. V., Théorie Analytique de la Chaleur. Vol. II, Gauthier-Villars, Paris, 1903

" The variations of density can be ignored except where they are multiplied by the acceleration of gravity in equation of motion for the vertical component of velocity vector."

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{1}{\rho_0} \cdot \frac{\partial \pi}{\partial x} \; ; \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{\rho_0} \cdot \frac{\partial \pi}{\partial y}$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -\frac{1}{\rho_0} \cdot \frac{\partial \pi}{\partial z} + \Gamma_0 \Theta$$

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = -\frac{K}{C_p} \left[\frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial y^2} + \frac{\partial^2 \Theta}{\partial z^2} \right]$$

$$\Gamma_0 = \alpha_0 g$$

$\alpha_0 g (\Theta - \Theta_0)/\rho_0 =$ **buoyancy**

$\pi =$ *pseudo*-pressure (Lagrange multiplier)

Oberbeck, A. 1879: Ueber die Wärmeleitung der Flssigkeiten bei Bercksichtigung der Strm̈ungen infolge von Temperaturdifferenzen. *Annalen der Physik* **243**, 271–292

Oberbeck, A. 1888: On the phenomena of motion in the atmosphere (first comm). *The Mechanics of the Earth's Atmosphere*, transl. by Cleveland Abbe. publ. Smithsonian Inst., Washington, 1891. pp. 176-187

2-component

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\frac{1}{\rho_0} \cdot \nabla \pi - \mathbf{g} \frac{\rho}{\rho_0}$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k_T \nabla^2 T + Q_T(\mathbf{x}, t)$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -k_S \nabla^2 T + Q_S(\mathbf{x}, t)$$

$$\rho = \rho_0 \left[1 - \alpha_0 (T - T_0) + \beta_0 (S - S_0) \right]$$

- The approximation involves EOS as well (exclude pressure, linearize)

Boussinesq approximation: Classical

Spiegel & Veronis, 1960:

(1) the fluctuation of density which appear with the advent of motion result principally from from thermal (as opposite to pressure) effects;

(2) in the equations for the rate of change of momentum and mass, density variations may be neglected except when they coupled to the gravitational acceleration in the buoyancy force;

Mihaljan, 1962

(3) calculate the substantial temperature change from the diffusion of heat only; [neglect heating/cooling by compression, as well as by viscous dissipation of mechanical energy]

Spiegel E. A. & G. Veronis, 1960: On the Boussinesq approximation for a compressible fluid. *Astrophys. J.*, **131**, 442-447.

Mihaljan, J. M., 1962: A rigorous exposition of the Boussinesq approximations applicable to a thin layer of fluid. *Astrophys. J.*, **136**, 1126-1133.

non-dimensional control parameters:

$$\epsilon_1 \equiv \alpha_0 \Theta$$

$$\epsilon_2 \equiv \frac{k_0^2}{C_0 L^2 \Theta}$$

$\sigma = \nu_0 / k_0$ Prandtl number

$R = \dfrac{g \alpha_0 \Theta L^3}{\nu_0 k_0}$ Rayleigh number

Boussinesq equations are derived as leading-order asymptotic expansion of Navier–Stokes with respect to ϵ_1, ϵ_2

Zeytonian, (2003, 100-year anniversary of Boussinesq approximation):
Formal asymptotic limit of Navier–Stokes equations if

$$\begin{aligned} c &\rightarrow \infty \qquad \rightarrow \quad \text{incompressible} \\ \rho' = \rho - \rho_0 &\rightarrow 0 \\ g &\rightarrow \infty \end{aligned}$$

however

$$g\rho'/\rho_0 \quad \text{remains finite}$$

small parameters

$$\epsilon_1 = \frac{gh}{c^2} \qquad \epsilon_2 = \frac{\rho'}{\rho_0}$$

- Note: $c \rightarrow \infty$ and $g \rightarrow \infty$ also leads to rigid-lid, since $p_{\text{surf}} = g\zeta$ remains finite

Batchelor, G. K., 1967;

McDougall, Greatbatch, & Lu, 2002;

Losch, Adcroft, & Campin, 2004;

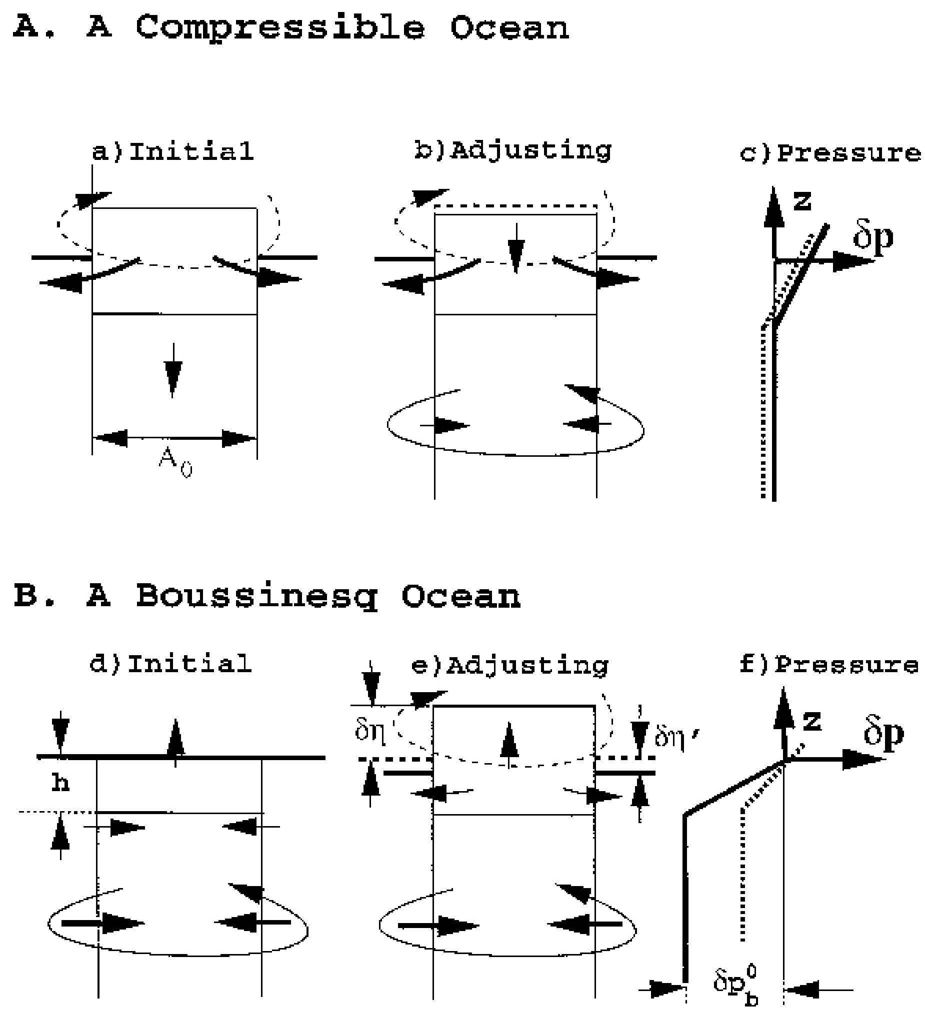
Boussinesq approximation: Consequences

- *inertial* and *gravitational* masses are no longer equivalent
- *some say*: in the absence of external forces Boussinesq models conserve volume, while non-Boussinesq mass. *More accurately*: if EOS is linear Boussinesq model conserves *both* volume and mass, it is the nonlinearity of EOS which brings the distinction.
- Kinetic Energy (KE) is per unit volume, not per unit mass, KE+PE is conserved as long as EOS is linear (up until Young, 2010)
- angular momentum balance is per unit volume
- pressure is no longer related with state variables via EOS ⇒ elimination of acoustic waves
- elastic (compression) energy is excluded from energy balance
- pressure becomes *Lagrange multiplier* (as in incompressible flows just to keep it non-divergent)
- we owe our ability to split pressure $P = P_{\text{free surface}} + P_{\text{baroclinic}} + Q_{\text{non-hydrostatic}}$ to Boussinesq approximation
- reversal of responses to external heating/cooling: non-Boussinesq model increases volume, Boussinesq decreases mass in response to surface heating, Greatbatch, 1994; Mellor & Ezer, 1995; Huang & Jin, 2002; Griffies, 2004(book)
- No steric effect

Boussinesq vs. non-Boussinesq: response to heating from surface

from Huang & Jin, 2002, 2-layer fluid

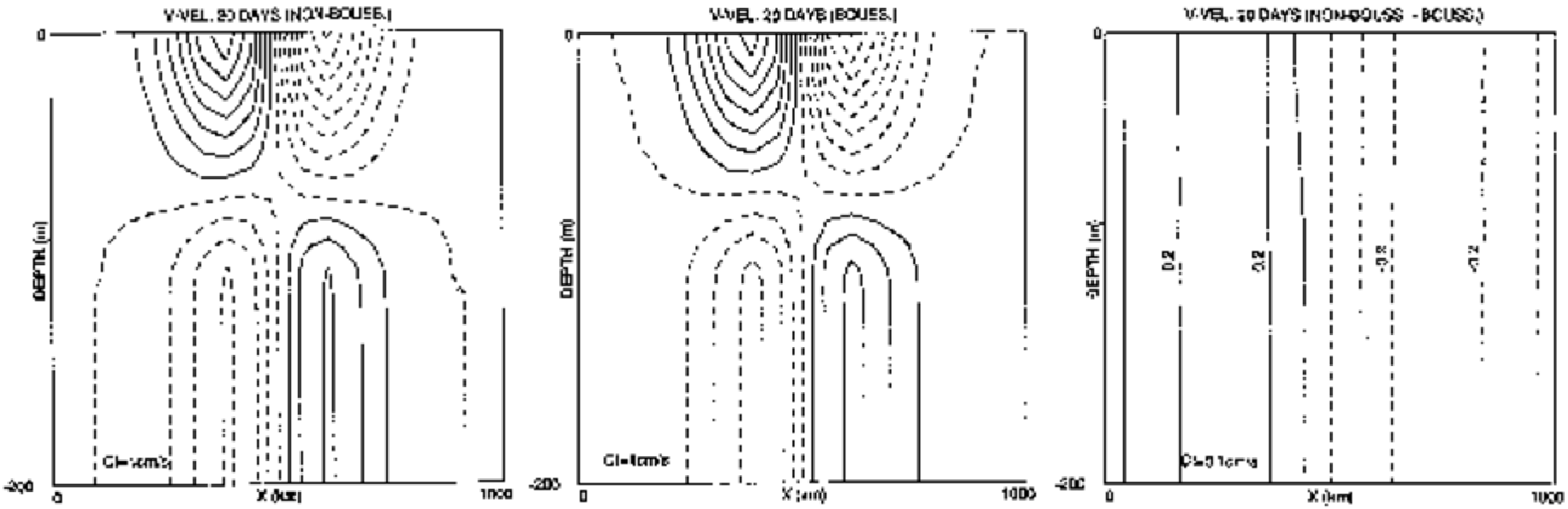
- *non-Boussinesq*
  immediate bulging of free surface
  ⇒ outward pressure-gradient force in upper layer; none in lower;
  ⇒ outward motion in upper layer, geostrophic adjustment;
  ⇒ decrease of pressure in both layers
  ⇒ inward motion in lower layer, geostrophic adjustment;
  ⇒ lifting thermocline
- *Boussinesq*
  immediate decrease of mass in upper layer;
  ⇒ inward pressure-gradient in lower layer; none in upper
  ⇒ inward motion in lower layer, geostrophic adjustment;
  ⇒ lifting thermocline and free surface;
  ⇒ outward pressure-gradient force in upper layer;
  ⇒ outward motion in upper layer, geostrophic adjustment



- The net outcome is similar in both cases: baroclinic flow, – cyclonic in upper-, anti-cyclonic in lower layer; net angular momentum conservation

Boussinesq vs. non-Boussinesq: response to heating from surface

from Mellor & Ezer, 1995



non-Boussinesq Boussinesq non-Boussinesq - Boussinesq

Field shown: tangential velocity component; contour interval 1cm for 2 left panels; 0.1 for difference. Max value 7cm/sec near surface; 3.5 near bottom.

The barotropic flow is cyclonic for Boussinesq case, anticyclonic (very weak) for non-Boussinesq

- The smallness of the difference, and ultimately the accuracy of Boussinesq model response relies on *stiffness* of the barotropic mode

Boussinesq approximation: softening incompressibility assumption

Atmospheric:
$\rho \to \rho_0$ in inertial terms is too crude, however
EOS of ideal gas $\Rightarrow$ simple

- *anelastic*: Ogura & Phillips, 1962

$$\nabla \cdot (\overline{\rho}\mathbf{u}) = 0$$

  where $\overline{\rho} = \overline{\rho}(z)$

$$\frac{w}{\overline{\rho}} \cdot \frac{\mathrm{d}\overline{\rho}}{\mathrm{d}z} + \nabla \cdot \mathbf{u} = 0$$

- Durran, 1989: Improving anelastic approximation. *J. Atmos. Sci.*, **46**, 1453-1461

$$\frac{1}{\rho^*} \cdot \frac{\mathrm{D}\rho^*}{\mathrm{D}t} + \nabla \cdot \mathbf{u} = 0$$

  where $\rho^* = \overline{\rho}\frac{\overline{\Theta}}{\Theta} = \overline{\rho}\frac{\overline{T}}{T}\left(\frac{p}{\overline{p}}\right)^{R/C_p}$

- Ingersoll, 2005
- energetic consistency?
- *modified Boussinesq*
  Durran, D. R., & A. Arakawa, 2007: Generalizing the Boussinesq approximation to stratified compressible flow. *Comptes Rendus Mecanique* **335**, 655-664

Oceanic: inertial $\rho \to \rho_0$ acceptable, but...

- complete exclusion pressure from EOS $\Rightarrow$ loss of thermobaric effect $\Rightarrow$ not acceptable
- $T \to \Theta$ (heating due to compression)
- ~40 years of using full (UNESCO-type since 198x) seawater EOS in Boussinesq models

Millero, Chen, Bradshaw, & Schleighter, 1980: A new high-pressure equation of state for seawater. *Deep. Sea Res.*, **27A** , 255-264.

Fofonoff & Millard, 1983: Algorithms for computation of fundamental properties of seawater. *Unesco Tech. Papers in Marine Sci.*, **44**, 53 pp.

- seawater EOS is kept outside Boussinesq vs. non-Boussinesq consideration
- KE+PE conservation for nonlinear EOS?
- using dynamic vs. reference ($-\rho_0 g z$) pressure in EOS $\Rightarrow$ unsettled
- free surface and barotropic-baroclinic mode splitting: existing splitting algorithms are not compatible with compressible EOS

Boussinesq approximation: timeline in ocean modeling

- Garrett & McDougall, 1992: criticize Boussinesq approximation for $\nabla\cdot\mathbf{u}=0$ using statistical, subgrid-scale parameterization arguments

- Davis, 1994: negate the above

- Dewar, Hsueh, McDougall, & Yuan, 1998: sensitivity to using full dynamic vs. reference pressure ($-\rho_0 gz$, basically depth) in EOS. Advocated full pressure regardless of whether the model is Boussinesq or not;

- Dukowicz, 2001: noted that Dewar et. al. 1998 error is self-canceling in a non-Boussinesq model, and is artifact of using full EOS with full pressure in Boussinesq. Proposed remedy by modifying EOS in Boussinesq model – "stiffening" of EOS;

- McDougall, Greatbatch, & Lu., 2002: acknowledged that Davis, 1994 is correct, Garrett & McDougall, 1992 is wrong. Proposed an alternative form of non-Boussinesq Eqs. written in terms of renormalized velocity $\bar{\tilde{\mathbf{u}}}=\overline{\rho\mathbf{u}}/\rho_0$ and a Boussinesq-like approximation derived from it. Advance the idea of re-interpreting output from Boussinesq model as non-Boussinesq. Rejected Dukowicz, 2001 approach as unnecessary and "entrenchment of Boussinesq conundrum"

$$\left.\begin{array}{rcl}
\dfrac{\partial}{\partial t}\left(\dfrac{\bar\rho}{\rho_0}\right) & + & \nabla\cdot\bar{\tilde{\mathbf{u}}}=0 \\
\dfrac{\partial}{\partial t}\left(\dfrac{\bar\rho}{\rho_0}\overline{C}^{\rho}\right) & + & \nabla\cdot\left(\bar{\tilde{\mathbf{u}}}\overline{C}^{\rho}\right)=\nabla\cdot\left(K\nabla\overline{C}^{\rho}\right) \\
\dfrac{\partial\bar{\tilde{\mathbf{u}}}}{\partial t}+\nabla\cdot\left(\dfrac{\rho_0}{\bar\rho}\bar{\tilde{\mathbf{u}}}\bar{\tilde{\mathbf{u}}}\right) & + & 2\Omega\times\bar{\tilde{\mathbf{u}}}= \\
-\dfrac{1}{\rho_0}\nabla\bar p & - & \mathbf{k}g\dfrac{\bar\rho}{\rho_0}+\nabla\cdot\left(A\nabla\dfrac{\rho_0}{\bar\rho}\bar{\tilde{\mathbf{u}}}\right)
\end{array}\right\}\rightarrow
\begin{array}{rcl}
 & & \nabla\cdot\bar{\tilde{\mathbf{u}}}\approx 0 \\
\dfrac{\partial\overline{C}^{\rho}}{\partial t} & + & \nabla\cdot\left(\bar{\tilde{\mathbf{u}}}\overline{C}^{\rho}\right)\approx\nabla\cdot\left(K\nabla\overline{C}^{\rho}\right) \\
\dfrac{\partial\bar{\tilde{\mathbf{u}}}}{\partial t} & + & \nabla\cdot\left(\bar{\tilde{\mathbf{u}}}\bar{\tilde{\mathbf{u}}}\right)+2\Omega\times\bar{\tilde{\mathbf{u}}}\approx \\
 & & -\dfrac{1}{\rho_0}\nabla\bar p-\mathbf{k}g\dfrac{\bar\rho}{\rho_0}+\nabla\cdot\left(A\nabla\bar{\tilde{\mathbf{u}}}\right)
\end{array}$$

[Note: (*i*) steady, geostrophic hydrostatic Eqs. are fully non-Boussinesq; (*ii*) desturbs mutual scaling between advection and Coriolis terms ⇒ interferes with PV theorem]

...if Joseph Valentin Boussinesq (1842–1929) and Anton Oberbeck (1846–1900) were live long enough, they would start with

$$\rho_{\text{EOS}}(P,S,\Theta) = \rho_0^{\bullet} + \frac{P}{c_0^2} + \frac{P^2}{c_2} + ... + \beta_0 S \quad - \quad \alpha_0\left(\Theta - \Theta_0\right) - \underbrace{\gamma_0 P\left(\Theta - \Theta_0\right)}_{\text{thermobaric}}$$

$$- \quad \underbrace{\delta_0\left(\Theta - \Theta_0\right)^2 - \nu_0 S\left(\Theta - \Theta_0\right)^2}_{\text{cabbeling}}$$

$$+ \quad \epsilon_0 \cdot S^{3/2} + ...$$

into

$$\text{buoyancy} = -\frac{g\rho^{\bullet\prime}}{\rho_0^{\bullet}} = A_0\left(\Theta - \Theta_0\right)\left[1 + \Gamma_0 P - D_0\left(\Theta - \Theta_0\right) + ...\right]$$

$$-B_0\left(S - S_0\right)\left[1 + E_0 S^{1/2} + ...\right]$$

Boussinesq approximation: oceanographic

Young, 2010: "seawater Boussinesq approximation"

(1) The exact density $\rho(\mathbf{x}, t)$ in the inertial terms of the momentum equations is replaced by constant reference density ρ_0,

(2) The mass conservation equation is approximated by $\nabla \cdot \mathbf{u}$, and

(3) the full EOS relates the buoyancy of seawater to temperature, salinity, and **an approximate pressure $P_0 = \rho_0 g_0 Z$ where $Z = Z(\mathbf{x})$ is the geopotential height** (i.e., the gravitational-centrifugal potential divided by the standard gravity at mean sea level, g_0).

- regains KE+PE energetic consistency
- contradics Dewar et. al. (1998) recommendations

EOS Compressibility effects in Boussinesq model

Example of internal inconsistency of Boussinesq model with compressible EOS: Assume spatially uniform $\Theta, S = \text{const}$, perturbed free surface, and hydrostatic balance,

$$\rho = \rho(P) = \rho\Big|_{P=0} + P/c^2, \qquad \partial_z P = -g\rho, \quad \text{along with} \quad P\Big|_{z=\zeta} = 0$$

which remaps $P \leftrightarrow z$ as

$$\rho = \rho\Big|_{z=\zeta} \exp\left\{g\frac{\zeta - z}{c^2}\right\} \approx \rho\Big|_{z=\zeta} + g\rho_0\frac{\zeta - z}{c^2}$$

[c is speed of sound, $1/c^2 = \partial\rho/\partial P$, and for simplicity we assume smallness $g|z|/c^2 \ll 1$ which is not principal] The acceleration created PGF due to perturbation in free surface is then

$$-\frac{1}{\rho_0}\nabla_x P = -\frac{g\ \rho|_{z=\zeta}}{\rho_0}\cdot\nabla_x\int_z^\zeta \exp\left\{g\frac{\zeta - z'}{c^2}\right\}dz' \approx -g\left[\frac{\rho|_{z=\zeta}}{\rho_0} + \frac{g(\zeta - z)}{c^2}\right]\nabla_x\zeta$$

increases with depth.

non-Boussinesq answer: acceleration is $-g\nabla_x\zeta$ **independent** from depth; this is **an exact** result, even if $g|z|/c^2$ is not small, and even in the case when $\rho = \rho(P)$ is nonlinear. **Both** ρ and $\nabla_x P$ contain multiplier

$$r(z) = \exp\left\{g\frac{\zeta - z}{c^2}\right\}$$

which cancels out when computing acceleration $-(1/\rho)\nabla_x P$
Boussinesq approximation retains $r(z)$ in one place but neglects it in the other, resulting in spurious vertical shear in PGF acceleration.

How does this affect ROMS?

- in computation of pressure gradient force: if EOS of JM95 (UNESCO-type) is used, the resultant r.h.s. for 3D momenta fully contains spurious vertical shear
- when computing

$$\overline{\rho} = \frac{1}{D}\int_{-h}^{\zeta} \rho \, dz \quad \text{and} \quad \rho_* = \frac{2}{D^2}\int_{-h}^{\zeta} \left\{ \int_{z}^{\zeta} \rho \, dz' \right\} dz \, , \qquad D = h + \zeta$$

  for the use in barotropic-mode pressure gradient, the resultant $\overline{\rho}$ and ρ_* are related as it would be for a stratified water column, when in fact, physically there is no stratification [e.g., $\rho = \rho(P)$ case];

- EOS is computed in *slow time*, but contains traces of free surface signal, which is kept unchanged during fast-time stepping $\Rightarrow$ contribution to mode splitting error. The contribution is small, but free-surface field available to EOS is comes from previous time step, it results in effectively Forward Euler stepping for these terms. This kind of instability was first observed in POM and reported by Robinson, Padman, & Levine, 2001: A correction to the baroclinic pressure gradient term in the Princeton ocean model. *J. Atmos. Ocean. Technol.*, **18**, pp. 1068-1075 [although they did not classify it as mode-splitting instability]. Their proposed remedy is to suppress compressibility effects in EOS altogether.
- Griffies advocates abandoning Boussinesq approximation, incl. the use of *in situ* pressure inside EOS. Although this eliminates spurious shear, it does not fix mode splitting (one must somehow exclude influence of free surface in EOS, which contradicts the idea of *in situ* pressure; or redesign barotropic mode; implicit stepping for free-surface is immune to this because it is too dissipative). Something remains to be done about *adiabatic differencing* (discussed below)

Three are 4 reasons why a Boussinesq ocean model needs EOS:

- computation of pressure gradient force;
- evaluation of stability of stratification as well as stability of external thermodynamic forcing (buoyancy flux) needed for mixing and planetary boundary layer parameterization;
- computation of slopes of neutral surfaces need by horizontal (along isopycnals) diffusion
- computing of $\overline{\rho}$ (vertically averaged density) and ρ_* (normalized vertically averaged pressure),which participate in barotropic–baroclinic mode splitting.

the role of EOS is to translate gradients of Θ, S into gradients of density

in situ **density is not needed**

PGF in sigma-coordinates needs

$$\mathcal{J}_{x,s}(\rho, z) = -\alpha \cdot \mathcal{J}_{x,s}(\Theta, z) + \beta \cdot \mathcal{J}_{x,s}(S, z)$$

where $\alpha = \alpha(\Theta, S, P) = -\left.\frac{\partial \rho}{\partial \Theta}\right|_{S,P=\text{const}}$ $\beta = \beta(\Theta, S, P) = \left.\frac{\partial \rho}{\partial S}\right|_{\Theta,P=\text{const}}$

alternatively, if

$$\rho = \rho_1^{(0)} + \rho_1'(\Theta, S) + \sum_{m=1}^{n} \left(q_m^{(0)} + q_m'(\Theta, S)\right) \cdot |z|^m$$

then $\rho_1^{(0)}$, $q_m^{(0)}$-terms are all out:

$$\mathcal{J}_{x,s}(\rho, z) = \mathcal{J}_{x,s}(\rho_1', z) + \sum_{m=1}^{n} \mathcal{J}_{x,s}(q_m', z) \cdot |z|^m$$

adiabatic differencing

$$\Delta\rho'^{(\text{ad})}_{i+\frac{1}{2},j,k} = \rho'_{1\,i+1,j,k} - \rho'_{1\,i,j,k} + \sum_{m=1}^{n} \left(q'_{m\,i+1,j,k} - q'_{m\,i,j,k}\right) \left|\frac{z_{i+1,j,k} + z_{i,j,k}}{2}\right|^m$$

the two adjacent adiabatic differences are averaged using harmonic mean, and (if needed) the compressible part is computed and added separately,

$$d_{i,j,k} \equiv \left.\frac{\partial \rho}{\partial \xi}\right|_{i,j,k} = \frac{1}{\Delta\xi} \cdot \frac{2\Delta\rho'^{(\text{ad})}_{i+\frac{1}{2},j,k} \cdot \Delta\rho'^{(\text{ad})}_{i-\frac{1}{2},j,k}}{\Delta\rho'^{(\text{ad})}_{i+\frac{1}{2},j,k} + \Delta\rho'^{(\text{ad})}_{i-\frac{1}{2},j,k}} + \left(q'_{1\,i,j,k} + 2q'_{2\,i,j,k} z_{i,j,k} + ...\right) \left.\frac{\partial z}{\partial \xi}\right|_{i,j,k},$$

the above guarantees monotonic stratification of cubic polynomial interpolant for density critical for PGF static stability; simple differencing does not

Boussinesq approximation and stiffening of EOS

Dukowicz, 2001 idea [also Sun, Bleck, Rooth, Dukowicz, Chassignet, & Killworth, 1999]: most variation of *in situ* density occur due to changes in pressure, and a much smaller fraction due to changes in Θ, S, hence

$$\rho = r(P) \cdot \rho^{\bullet}(\Theta, S, P)$$

where $r(P)$ is a universal function (*does not depend on local* Θ, S) which can be chosen to "absorb" most of variation of density due to pressure.

- $\rho^{\bullet}(\Theta, S, P)$ fully retains thermobaric and cabbeling effects.

- variation of Θ and S decrease with depth (survey of Levitus data)

- allows a self-consistent ($\Rightarrow$ more accurate) remapping $r(P) \to r(z)$ and $\rho^{\bullet}(\Theta, S, P) \to \rho^{\bullet}(\Theta, S, z)$ as an alternative to bulk pressure $P = g\rho_0|z|$ inside EOS

- substitution of $r(P) \cdot \rho^{\bullet}(\Theta, S, P)$ into non-Boussinesq equations shows tendency of $r(P)$ to cancel out (exactly or approximately) in all terms which depend on density: e.g., it removes spurious barotropic shear; $r(z)$ commutes with density Jacobian operator,

  $$\mathcal{J}_{x,s}\Big(r(z) \cdot \rho^{\bullet}, z\Big) = r(z) \cdot \mathcal{J}_{x,s}\Big(\rho^{\bullet}, z\Big)\,,$$

  and BVF computed using a Boussinesq-like rule

  $$N^2 = -\frac{q}{\rho_0^{\bullet}} \left[\frac{\partial \rho^{\bullet}}{\partial \Theta}\bigg|_{S,z=\text{const}} \frac{\partial \Theta}{\partial z} + \frac{\partial \rho^{\bullet}}{\partial S}\bigg|_{\Theta,z=\text{const}} \frac{\partial S}{\partial z} \right]$$

  is closer to its non-Boussinesq counterpart than in the case of standard Boussinesq approximation [$\rho_0^{\bullet}$ is a constant similar to Boussinesq reference density, but representing $\rho^{\bullet}$ instead of *in situ* density]

Practical "stiffened" EOS for ROMS

From EOS of Jackett & McDougall, 1995,

$$\rho(\Theta,S,z) = \frac{\rho_1(\Theta,S)}{1-0.1\cdot z\,/[K_{00}+K_0(\Theta,S)+K_1(\Theta,S)\cdot z+K_2(T,S)\cdot z^2]}$$

chose

$$r(z) = \frac{1}{1-0.1z/K^{\text{ref}}(z)} \quad \text{with} \quad K^{\text{ref}}(z) = K_{00}+K_0^{\text{ref}}+K_1^{\text{ref}}z+K_2^{\text{ref}}z^2$$

$$K_0^{\text{ref}} = K_0\left(\Theta^{\text{ref}},S^{\text{ref}}\right), \qquad K_1^{\text{ref}} = K_1\left(\Theta^{\text{ref}},S^{\text{ref}}\right), \qquad K_2^{\text{ref}} = K_2\left(\Theta^{\text{ref}},S^{\text{ref}}\right).$$

Select representative *abyssal* values $\Theta^{\text{ref}} = 3.5$, $S = 34.5$, then $K_0^{\text{ref}} = 2924.921$, $K_1^{\text{ref}} = 0.34846939$, $K_2^{\text{ref}} = 0.145612\times10^{-5}$, and $\rho_1\left(\Theta^{\text{ref}},S^{\text{ref}}\right) = 1027.43879$.

The "stiffened" EOS

$$\rho^{\bullet\prime}(\Theta,S,z) = [\rho_0^{\bullet}+\rho_1'(\Theta,S)]\cdot\frac{1-0.1z/K^{\text{ref}}(z)}{1-0.1z/K(\Theta,S,z)}-\rho_0^{\bullet}$$

$\rho^{\bullet\prime}$ is perturbation of $\rho^{\bullet}$ relatively to a constant reference value $\rho_0^{\bullet}$, for which $\rho_0^{\bullet} = \rho_1\left(\Theta^{\text{ref}},S^{\text{ref}}\right)$ is the natural choice. Cancelling large terms,

$$\begin{aligned}
\rho^{\bullet\prime}(\Theta,S,z) &= \rho_1'(\Theta,S)+0.1z\cdot\frac{\rho_0^{\bullet}+\rho_1'(\Theta,S)}{K_{00}+K_0^{\text{ref}}+K_1^{\text{ref}}z+K_2^{\text{ref}}z^2}\times \\
&\times \quad \frac{K_0^{\text{ref}}-K_0(\Theta,S)+\left(K_1^{\text{ref}}-K_1(\Theta,S)\right)\cdot z+\left(K_2^{\text{ref}}-K_2(\Theta,S)\right)\cdot z^2}{K_{00}+K_0(\Theta,S)+(K_1(\Theta,S)-0.1)\,z+K_2(\Theta,S)z^2} \\
&= \rho_1'(\Theta,S)+\widehat{q}'(\Theta,S,z)\cdot z
\end{aligned}$$

so far without any approximation.

Property $\rho^{\bullet\prime}\left(\Theta^{\text{ref}}, S^{\text{ref}}, z\right) \equiv 0$, and, similarly, $\widehat{q}^{\prime}\left(\Theta^{\text{ref}}, S^{\text{ref}}, z\right) \equiv 0$ regardless of z ensures that variation of $\rho^{\bullet\prime}$ is expected to be small, and decrease with depth because variation of Θ and S also decrease.

Already close to the desired form, but $\widehat{q}^{\prime}(\Theta, S, z)$ still explicitly depends on z, although the dependency is weak in comparison with the original JM95. Taylor expansion for powers of z yields

$$\rho^{\bullet\prime}(\Theta, S, z) = \rho_1^{\prime}(\Theta, S) + q_1^{\prime}(\Theta, S) \cdot z$$

where

$$q_1^{\prime}(\Theta, S) = 0.1 \cdot \left[\rho_0 + \rho_1^{\prime}(\Theta, S)\right] \cdot \frac{K_0^{\text{ref}} - K_0(\Theta, S)}{\left[\left(K_{00} + K_0(\Theta, S)\right) \cdot \left(K_{00} + K_0^{\text{ref}}\right)\right]}$$

with $q_1^{\prime}$ does not depend on z.

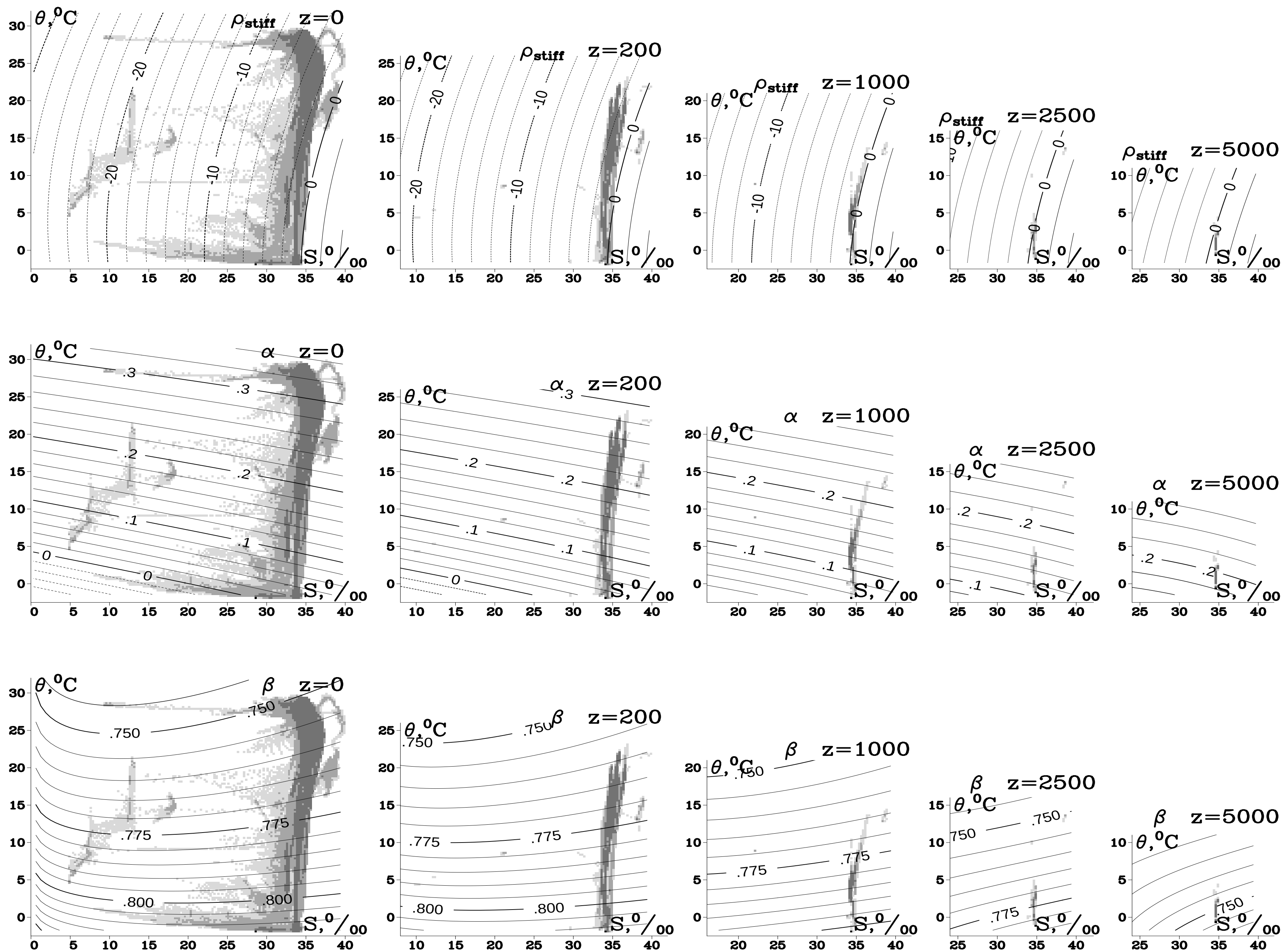
This involves **an approximation** – discards all coefficients associated with K_1 and K_2 terms (hence 14 out of 26 in the original JM95 bulk secant modulus). Naturally, this **raises concern about the accuracy**.

The final version is,

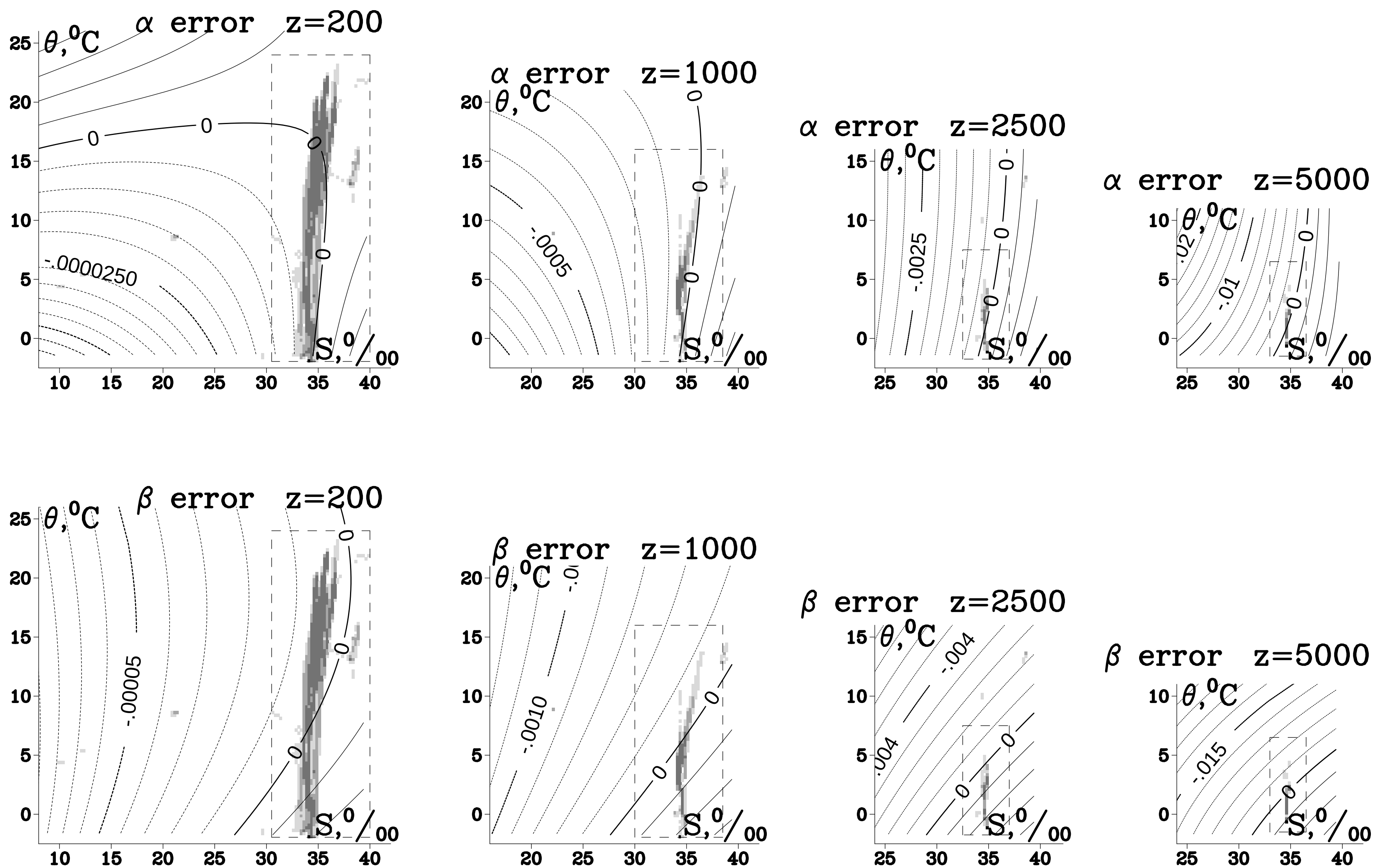
$$\rho^{\bullet\prime}(\Theta, S, z) = \rho_1^{\prime}(\Theta, S) + q_1^{\prime}(\Theta, S) \cdot z\,(1 - \gamma z)$$

with $\gamma = 1.72 \times 10^{-5}$ for Θ^{ref} and S^{ref} from above, γ is just a constant.

"Stiffened" EOS in ROMS: Properties

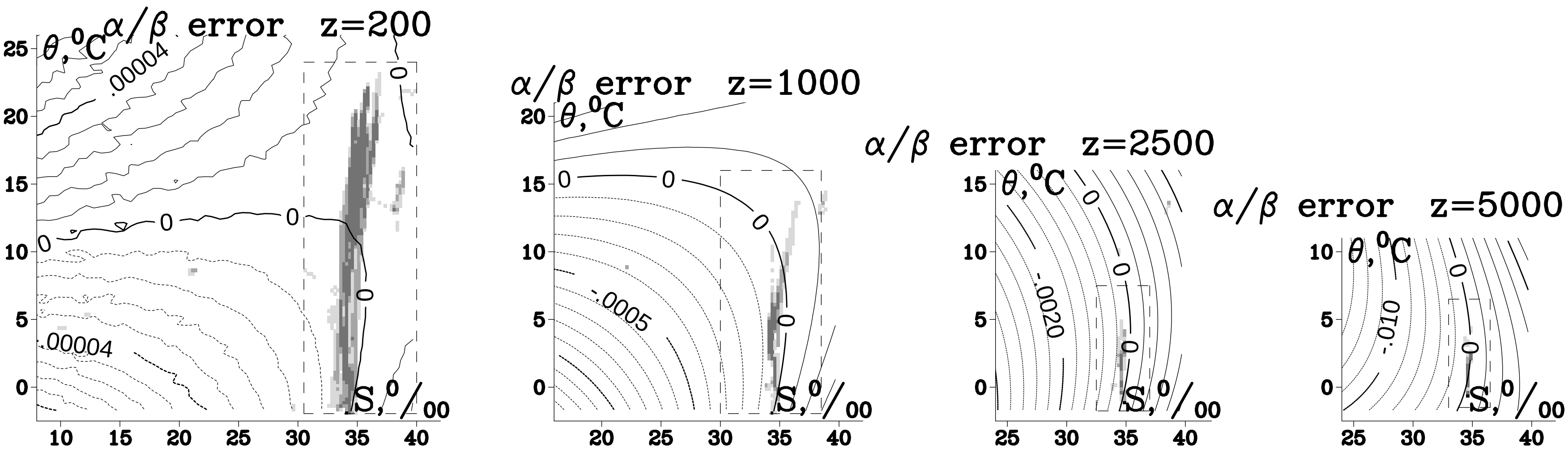


"Stiffened" EOS in ROMS: Accuracy



- errors are relative to Jackett & McDougall (1995) and non-Boussinesq

"Stiffened" EOS in ROMS: Accuracy



- zero-error alignment with available data
- accuracy for α/β is more demanding than for α and β separately

"Stiffened" EOS in ROMS: Summary

- Follows Dukowicz, 2001, except in choosing constant Θ, S reference to construct $r(P) \rightarrow r(z)$, rather than globally averaged profile from Levitus. This is to facilitate adiabatic differencing critical for PGF in ROMS (z-coordinate models do not care)
- fully retains thermobaric and cabbeling effects
- If Boussinesq approximation is applied, **it must be applied to EOS as well**
- in comparison with 2003 PGF study allows to align zero-error point on Θ, S plane with the desired location. Approximately one order of magnitude more accurate.
- adiabatic derivatives normalized by stiffened reference density, e.g.,

$$N^2 = -\frac{q}{\rho_0^\bullet}\left[\left.\frac{\partial \rho^\bullet}{\partial \Theta}\right|_{S,z=\text{const}} \frac{\partial \Theta}{\partial z} + \left.\frac{\partial \rho^\bullet}{\partial S}\right|_{\Theta,z=\text{const}} \frac{\partial S}{\partial z}\right]$$

  are close to that from non-Boussinesq model
- **PGF scheme and KPP are updated to accommodate the change**
- removes most (up to $\sim 90\%$) of Boussinesq approximation errors; replaces $\rho_0 = const$ reference with $\rho_0^\bullet r(z)$, which closer to reality
- the reason why it works well is because $r(z)$ in *integrable*, merely because $r \sim e^{gz/c^2} \sim 1 + gz/c^2$, with $gz/c^2 << 1$ so both density and pressure are multiplied by approximately the same factor (exactly the same in barotropic case), resulting in cancellation $r(z)$ (approx. or exact), and preserving semantics of Boussinesq code
- Eliminates mode splitting error in computing ρ^*, $\overline{\rho}$ without increase of code complexity (basically without any change in parts computing ρ^*, $\overline{\rho}$): now these two are purely baroclinic (no spurious stratification), and therefore assumption that they are kept constant during fast-time stepping is fully justified

Barotropic-Baroclinic Mode Splitting

Boussinesq ROMS discrete time stepping

3D continuity

$$\Delta z_k^{n+1} = \Delta z_k^n - \Delta t \cdot \left[\nabla_\perp (\Delta z_k \mathbf{u}_k) + w_{k+1/2} - w_{k-1/2}\right]^{n+1/2}$$

grid-box heights

$$\Delta z_k = \Delta z_k \left(\Delta z_k^{(0)}, \zeta\right) = \Delta z_k^{(0)} \left(1 + \frac{\langle\zeta\rangle}{h}\right)$$

free-surface

$$\langle\zeta\rangle^{n+1} = \langle\zeta\rangle^n - \Delta t \cdot \nabla_\perp \langle\langle\overline{\mathbf{U}}\rangle\rangle^{n+1/2}$$

where

$$\sum_{k=1}^{N} \Delta z_k = h + \langle\zeta\rangle\,; \quad \langle\langle\overline{\mathbf{U}}\rangle\rangle^{n+1/2} \equiv \langle\langle\overline{U}, \overline{V}\rangle\rangle^{n+1/2} = \left\{\sum_{k=1}^{N} \Delta z_k u_k\,,\ \sum_{k=1}^{N} \Delta z_k v_k\right\}^{n+1/2}$$

tracers

$$\Delta z_k^{n+1} q_k^{n+1} = \Delta z_k^n q_k^n - \Delta t \cdot \left[\nabla_\perp \left(q_k \Delta z_k \mathbf{u}_k\right) + q_{k+1/2} w_{k+1/2} - q_{k-1/2} w_{k-1/2}\right]^{n+1/2},$$

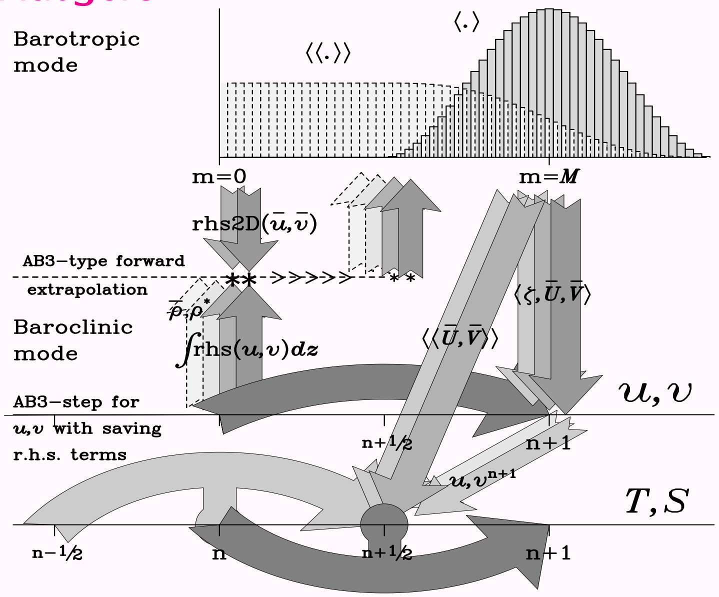
- **sequential** algorithm:

  r.h.s.3D $\to \langle\zeta\rangle^{n+1}, \langle\langle\overline{U}, \overline{V}\rangle\rangle^{n+1/2}, \langle\overline{U}, \overline{V}\rangle^{n+1} \to \Delta z_k \to u, v^{n+1} \to U, V \to W \to q^{n+1}$

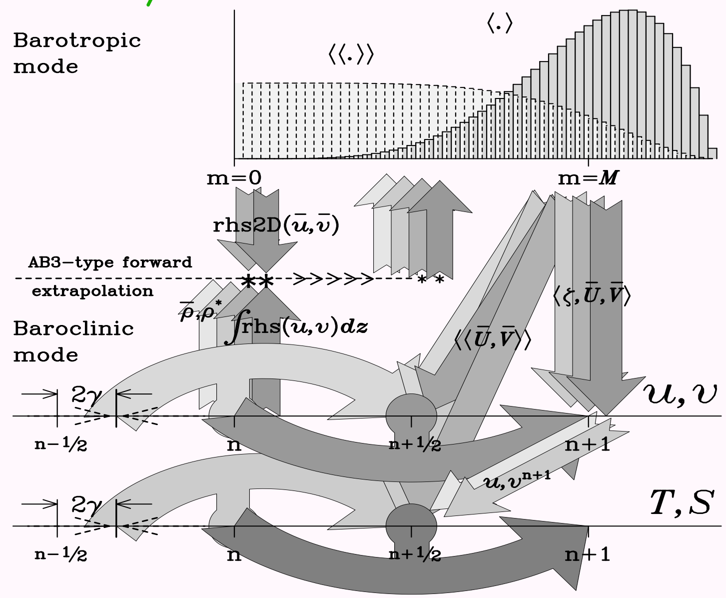
- **conservation and constancy**

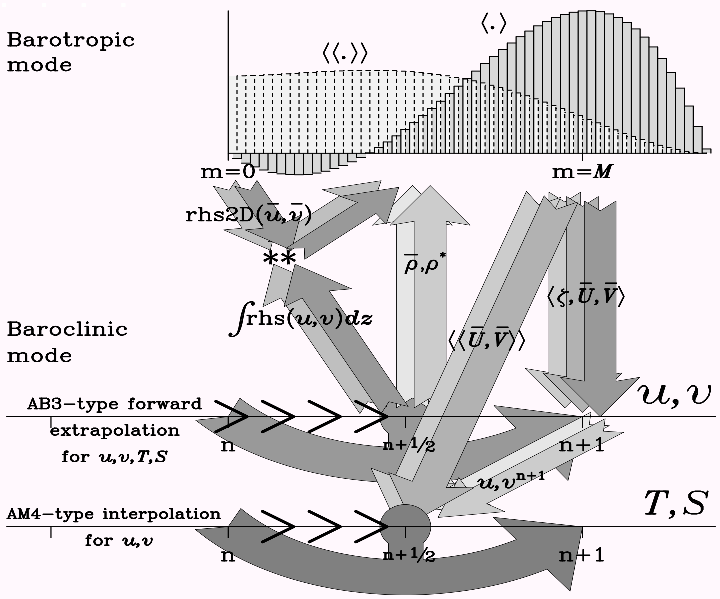
- EOS stays **entirely** within the slow mode

Rutgers

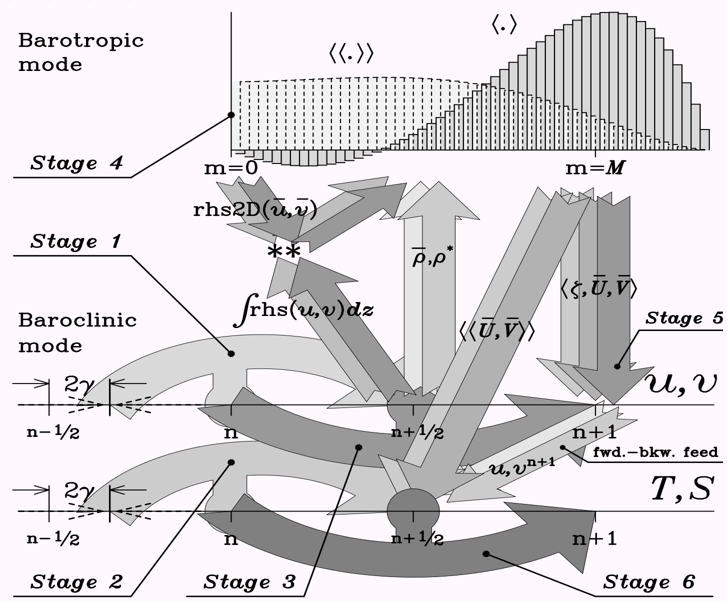


AGRIF/ old UCLA





Non-hydrostatic code prototype



UCLA (current)

non-Boussinesq code time stepping

mass conservation

$$\rho_k^{n+1}\Delta z_k^{n+1} = \rho_k^n \Delta z_k^n - \Delta t \cdot \left[\nabla_\perp\left(\rho_k \Delta z_k \mathbf{u}_k\right) + \rho_{k+1/2} w_{k+1/2} - \rho_{k-1/2} w_{k-1/2}\right]^{n+1/2}$$

grid-box heights $\quad \Delta z_k = \Delta z_k\left(\Delta z_k^{(0)}, \zeta\right) \quad$ (unchanged)

barotropic (free surface)

$$\overline{\rho}^{n+1}\left(h + \zeta^{n+1}\right) = \overline{\rho}^n\left(h + \zeta^n\right) - \Delta t \cdot \nabla_\perp \langle\langle \overline{\rho}\overline{\mathbf{U}}\rangle\rangle^{n+1/2}$$

where $\quad \overline{\rho} = \frac{1}{h+\zeta} \cdot \sum_{k=1}^{N} \Delta z_k \rho_k \quad$ and $\quad \sum_{k=1}^{N} \Delta z_k = h + \zeta$

$$\langle\langle \overline{\rho}\overline{\mathbf{U}}\rangle\rangle^{n+1/2} \equiv \langle\langle \overline{\rho}\overline{U},\, \overline{\rho}\overline{V}\rangle\rangle^{n+1/2} = \left\{\sum_{k=1}^{N} \rho_k \Delta z_k u_k\,,\ \sum_{k=1}^{N} \rho_k \Delta z_k v_k\right\}^{n+1/2}$$

tracers $q = \{\Theta, S\}$

$$\begin{aligned}\rho_k^{n+1}\Delta z_k^{n+1} q_k^{n+1} = \rho_k^n \Delta z_k^n q_k^n - \Delta t \cdot \Big[\nabla_\perp\left(q_k \cdot \rho_k \Delta z_k \mathbf{u}_k\right) \ &+ \ q_{k+1/2} \cdot \rho_{k+1/2} w_{k+1/2} \\ &- \ q_{k-1/2} \cdot \rho_{k-1/2} w_{k-1/2}\Big]^{n+1/2}\end{aligned}$$

$$\rho_k = \rho_{\mathrm{EOS}}(\Theta_k, S_k, P_k) \qquad \text{where} \quad P_k = g \sum_{k'=k+1}^{N} \rho_{k'} \Delta z_{k'} + \frac{1}{2} g \rho_k \Delta z_k\,.$$

- **cyclic dependency:** need ρ_k^{n+1} to compute $\Theta_k^{n+1}, S_k^{n+1}$, but ρ_k^{n+1} depends from them via EOS
- **non-splittable:** ζ^{n+1} needs $\overline{\rho}$, which depends on P_k, which depends on ζ via Δz_k

Two ways to break cyclic dependency:

- Greatbatch, R.J., Y. Lu, & Y. Cai, 2001: forward extrapolation of density

$$\rho^{(e)}{}_k^{n+1} = 2\rho_k^n - \rho_k^{n-1} \qquad \text{and} \quad \rho^{(e)}{}_k^{n} = 2\rho_k^{n-1} - \rho_k^{n-2} ,$$

  $\Rightarrow$ tracer conservation properties no longer exist in the original meaning

$$\sum \rho^{n+1} \Delta z^{n+1} q^{n+1} = \sum \rho^n \Delta z^n q^n ,$$

  while ρ^{n+1} are also related to $q^{n+1} = \left\{\Theta^{n+1}, S^{n+1}\right\}$ via EOS.

  $\Rightarrow$ splitting instability of barotropic mode

  Mellor & Ezer, 1995 used this approach

- de Szoeke & Samelson, 2002: showed duality (“isomorphism”) between *hydrostatic* z-coordinate Boussinesq and *hydrostatic* non-Boussinesq equations written in pressure-coordinates. Argued that Boussinesq approximation is redundant because nonhydrostatic alone removes acoustic waves, while Boussinesq offers no further simplification;

  $\Rightarrow$ commitment to hydrostatic modeling

  $\Rightarrow$ isomorphism is **incomplete**: does not apply to barotropic mode; breaks cyclic dependency, but does not help splittability

  modern, preferred approach

  splitting in existing non-Boussinesq codes either relies on $\rho - \rho_0 \ll \rho_0$ combined with heavy filtering; or use stiffened EOS: MICOM/HYCOM, HIM, Higdon (all).

Pressure-based-coordinate framework: $\rho_k \Delta z_k \equiv (1/g)\Delta p_k$ $\Leftarrow$ hydrostatic

$$\Delta p_k^{n+1} = \Delta p_k^n - \Delta t \cdot \left[\nabla_\perp \left(\Delta p_k \mathbf{u}_k\right) + \widetilde{\omega}_{k+1/2} - \widetilde{\omega}_{k-1/2}\right]^{n+1/2}$$

where, *somehow* $\Delta p_k^{n,n+1} = \Delta p_k \left(\Delta p_k^{(0)}, p_b^{n,n+1}\right)$ $\leftarrow$ vertical mapping

barotropic mode

$$p_b^{n+1} = p_b^n - \Delta t \cdot \nabla_\perp \langle\langle p_b \overline{\mathbf{u}} \rangle\rangle^{n+1/2}$$

bottom pressure

$$p_b = \sum_{k=1}^{N} \Delta p_k$$

$$\langle\langle p_b \overline{\mathbf{u}} \rangle\rangle^{n+1/2} \equiv \langle\langle p_b \overline{u},\, p_b \overline{v} \rangle\rangle^{n+1/2} = \left\{ \sum_{k=1}^{N} \Delta p_k u_k \,,\; \sum_{k=1}^{N} \Delta p_k v_k \right\}^{n+1/2}$$

tracers

$$\Delta p_k^{n+1} q_k^{n+1} = \Delta p_k^n q_k^n - \Delta t \cdot \left[\nabla_\perp \left(q_k \Delta p_k \mathbf{u}_k\right) + q_{k+1/2}\widetilde{\omega}_{k+1/2} - q_{k-1/2}\widetilde{\omega}_{k-1/2}\right]^{n+1/2}$$

- similar, **"isomorphic"** to Boussinesq, subject to replacement variables,

$$\begin{array}{rcl} \Delta p_k & \leftrightarrow & \Delta z_k \\ p_b & \leftrightarrow & D = h + \zeta \\ \widetilde{\omega}_{k+1/2} & \leftrightarrow & w_{k+1/2} \end{array}$$

- **sequential?**
- **is it splittable?**

- Sequential algorithm is possible for *non-split* system: Δz_k are not needed, except for pressure-gradient terms

- the role of EOS is reversed: it is now needed to compute grid-box heights,

$$\Delta z_k = \frac{\Delta p_k}{g} \cdot \alpha_{\mathrm{EOS}}(\Theta_k, S_k, P_k) \qquad \text{and free surface} \quad \zeta = \sum_{k=1}^{N} \Delta z_k - h\,,$$

  while pressure increments Δp_k and EOS pressure P_k are assumed to be known as long as p_b is known,

$$\Delta p_k = p_b \cdot \Delta z_k^{(0)}/h \qquad \text{where} \quad \Delta z_k^{(0)} = \Delta z^{(0)}(x,y) = z_{k+1/2}^{(0)}(x,y) - z_{k-1/2}^{(0)}(x,y)$$

  after which

$$P_k = \frac{P_{k+1/2} + P_{k-1/2}}{2} \quad \text{and} \quad P_{k-1/2} = \sum_{k'=k}^{N} \Delta p_{k'}$$

- overall similar to Boussinesq

- **splittable?**

- Knowing “bottom pressure” p_b is **not enough** to compute vertically-integrated pressure gradient term for barotropic mode: free-surface ζ is required. p_b and ζ are linked via EOS (hence in 3D) and because EOS is pressure-dependent, it also has “fast” dependencies

- **EOS becomes part of splitting algoritm**

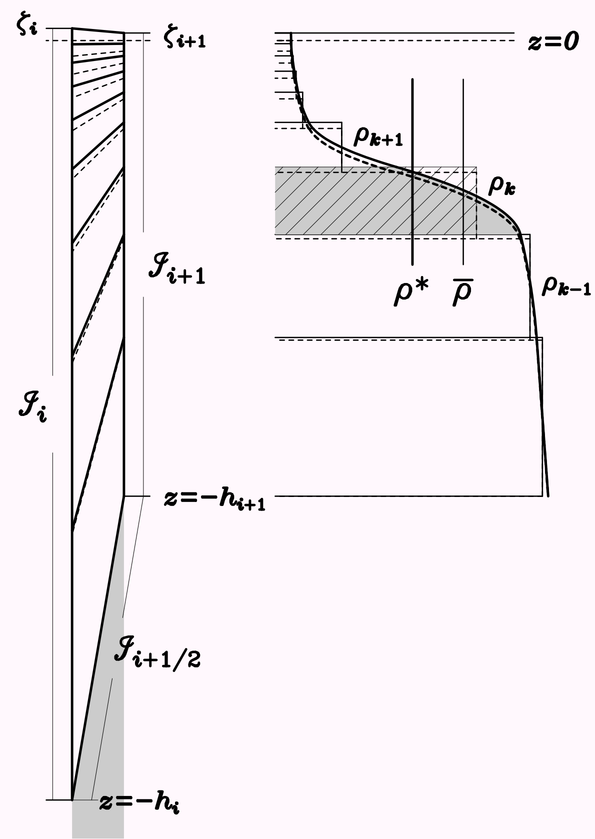
- last ditch effort to chicken out? “Stiffen” EOS in now *non-Boussinesq* code

Vertically-integrated non-Boussinesq momentum and mass-conservation

$$\partial_t\left(\overline{\rho}D\overline{\mathbf{u}}\right) + \underbrace{g\left[\nabla_\perp\left(\frac{\rho_* D^2}{2}\right) - \overline{\rho}D\nabla_\perp h\right]}_{=-\mathscr{F}} \;=\; \text{advection, Coriolis, dissipation, forcing}$$

$$\partial_t\left(\overline{\rho}D\right) + \nabla_\perp\left(\overline{\rho}D\overline{\mathbf{u}}\right) \;=\; \text{fresh-water flux}$$

where
$$\overline{\mathbf{u}} = \frac{1}{\overline{\rho}D}\int_{-h}^{\zeta}\rho\mathbf{u}\,\mathrm{d}z\,, \qquad \overline{\rho} = \frac{1}{D}\int_{-h}^{\zeta}\rho\,\mathrm{d}z\,,$$
$$D = h + \zeta\,, \qquad \rho_* = \frac{2}{D^2}\int_{-h}^{\zeta}\int_{z'}^{\zeta}\rho\,\mathrm{d}z'\mathrm{d}z$$



same, but in via **bottom pressure**, $p_b \equiv g\overline{\rho}D$,

$$\partial_t\left(p_b\overline{\mathbf{u}}\right) + \underbrace{g\left[\nabla_\perp\left(\frac{\rho_*}{\overline{\rho}}\cdot\frac{p_b(h+\zeta)}{2}\right) - p_b\nabla_\perp h\right]}_{=-\mathscr{F}} \;=\; \ldots$$

$$\partial_t p_b + \nabla_\perp\left(p_b\overline{\mathbf{u}}\right) \;=\; \ldots$$

[both p_b and ζ are present in $\mathscr{F}$]

Splitting $\mathscr{F} = -g\,[\ldots] =$ **“fast”** $+$ **“slow”**

- need $\zeta = \zeta(p_b)$ in “fast” time, i.e., via 2D fields only
- $\overline{\rho}, \rho_*$ slightly change – depend on p_b, ζ – because of EOS compressibility
- there is **stiff large-term cancellation** inside $\mathscr{F}$, especially on steep topography

c.f., SM2005 Boussinesq ROMS split:

- Boussinesq $\Rightarrow$ $\quad \overline{\rho} \rightarrow \rho_0 \quad$ everywhere except inside $\mathscr{F}$

- $\Rightarrow$ ζ becomes "fast" prognostic variable

$$\partial_t \left(\rho_0 D\right) + \nabla_\perp \left(\rho_0 D \overline{\mathbf{u}}\right) = ... \qquad \rightarrow \qquad \partial_t \zeta + \nabla_\perp \left(D \overline{\mathbf{u}}\right) = ...$$

- $\overline{\rho}, \rho_*$ independent from ζ (incompressible)

- large terms in $\mathscr{F}$ are canceled by hand

$$\partial_t \left(\rho_0 D \overline{\mathbf{u}}\right) + gD \left\{ \rho_* \nabla_\perp \zeta + \frac{D}{2} \nabla_\perp \rho_* + \left(\rho_* - \overline{\rho}\right) \nabla_\perp h \right\} = ...$$

- **all three issues go away**

Splitting non-Boussinesq: A "physicist's" approach

- extract $-gp_b\nabla_\perp\zeta$ from $\mathscr{F}$

$$\mathscr{F} = -g\left[\nabla_\perp\left(\frac{\rho_*}{\overline{\rho}}\cdot\frac{p_b(h+\zeta)}{2}\right) - p_b\nabla_\perp h\right] = -gp_b\nabla_\perp\zeta - g\nabla_\perp\left[\left(\frac{\rho_*}{\overline{\rho}}-1\right)\cdot\frac{p_b(h+\zeta)}{2}\right]$$
$$-\frac{1}{2}g\left[p_b\nabla_\perp(h+\zeta)-(h+\zeta)\nabla_\perp p_b\right]$$

- note that p_b is dominated by bulk part which does not change in time,

$$\rho = \rho_0 + \rho' \qquad \rho' \ll \rho_0$$
$$p_b = \rho_0 g h + p'_b \qquad p'_b \ll \rho_0 g h$$

after which

$$\mathscr{F} = -gp_b\nabla_\perp\zeta - g\nabla_\perp\left[\left(\frac{\rho_*}{\overline{\rho}}-1\right)\cdot\left(\frac{\rho_0 g h^2}{2} + \frac{p'_b+\rho_0 g\zeta}{2}h + \frac{p'_b\rho_0 g\zeta)}{2}\right)\right]$$
$$-\frac{1}{2}g\underbrace{\left[(p'_b-\rho_0 g\zeta)\nabla_\perp h - h\nabla_\perp(p'_b-\rho_0 g\zeta) + p'_b\nabla_\perp\zeta - \zeta\nabla_\perp p'_b\right]}_{=0,\quad \text{if} \quad \zeta=p'_b/(\rho_0 g)}$$

- **still need ζ via p'_b in "fast" time, use** $\zeta = \dfrac{p'_b}{\rho_0 g}$ hence

$$\partial_t\left(p_b\overline{\mathbf{u}}\right) + \underbrace{g\left(\rho_0 g h + p'_b\right)\nabla_\perp\frac{p'_b}{\rho_0 g}}_{\text{"fast"}} = -\underbrace{g\nabla_\perp\left[\left(\frac{\rho_*}{\overline{\rho}}-1\right)\cdot p'_b h\right]}_{\text{"mixed"}} - \underbrace{g\nabla_\perp\left[\left(\frac{\rho_*}{\overline{\rho}}-1\right)\frac{\rho_0 g h^2}{2}\right]}_{\text{baroclinic, "slow"}} + \ldots$$
$$\partial_t p'_b + \nabla_\perp\left(\left(\rho_0 g h + p'_b\right)\overline{\mathbf{u}}\right) = \ldots$$

Splitting non-Boussinesq: "physicist" continued

- why to extract $-gp_b \nabla_\perp \zeta$?

  it resembles PGF term of compressible, barotropic layer of fluid

- accuracy of this split relies on the following smallnesses:

$$\left| \frac{\rho_*}{\overline{\rho}} - 1 \right| \ll 1 \,, \qquad \left| \frac{\overline{\rho} - \rho_0}{\overline{\rho}} \cdot \nabla_\perp h \right| \ll 1 \,, \qquad \frac{gh}{c^2} \ll 1$$

- *c.f.*, "true" free surface

$$\zeta = D - h = \frac{p_b}{\overline{\rho} g} - h = \underbrace{h \left(\frac{\rho_0}{\overline{\rho}} - 1 \right)}_{\text{bias}} + \frac{p_b'}{\overline{\rho} g} \qquad \text{vs.} \qquad \zeta = \frac{p_b'}{\rho_0 g}$$

- influence of topography: splitting across large-terms which balance each other. To date all theoretical analysis of mode splitting comes from linearized surface-gravity – internal-wave normal-mode decomposition. All done in flat-bottom context.

- $\rho = \rho_{\text{EOS}} \left(..., \text{"fast"} \;\, p_b' \right)$ dependency ignored

- This is ... basically ... Boussinesq, but more exposed to sigma errors.

Mode splitting using incremental variables: Boussinesq first

introduce $\delta\zeta \equiv \zeta - \langle\zeta\rangle^n$, hence

$$\begin{array}{rcccl} \zeta & \rightarrow & \zeta + \delta\zeta & \equiv & \langle\zeta\rangle^n + \delta\zeta \\ D & \rightarrow & D + \delta\zeta & \equiv & h + \langle\zeta\rangle^n + \delta\zeta \end{array}$$

insert into $\mathscr{F}$ and expose all terms containing $\delta\zeta$,

$$\frac{\partial}{\partial t}\Big(\rho_0 (D+\delta\zeta)\overline{\mathbf{u}}\Big) - \mathscr{F} + g\,(D+\delta\zeta)\,\nabla_\perp\,(\rho_*\delta\zeta) + g\rho_*\delta\zeta\nabla_\perp\zeta$$
$$-g\frac{\delta\zeta^2}{2}\nabla_\perp\rho_* + g\,(\rho_* - \overline{\rho})\,\delta\zeta\nabla_\perp h \quad = \ldots$$
$$\frac{\partial}{\partial t}\delta\zeta + \nabla_\perp\Big((D+\delta\zeta)\overline{\mathbf{u}}\Big) = \ldots$$

where $\delta\zeta$ and $\overline{\mathbf{u}}$ are **the only** variables which are changing in "fast" time.
All others: $\mathscr{F}$, D, $\overline{\rho}$, ρ_*,... are computed by 3D and **retain** their n-th status.

works as follows

$$\underbrace{\begin{pmatrix} \delta\zeta = 0 \\ \\ \overline{\mathbf{u}} = \langle\overline{\mathbf{u}}\rangle^n \end{pmatrix}}_{\text{initial at step}\; n} \rightarrow \underbrace{\begin{pmatrix} \langle\delta\zeta\rangle^{n+1} \\ \langle(D+\delta\zeta)\overline{\mathbf{u}}\rangle^{n+1} \\ \langle\!\langle(D+\delta\zeta)\overline{\mathbf{u}}\rangle\!\rangle^{n+1/2} \end{pmatrix}}_{\text{fast-time stepping}} \rightarrow \underbrace{\begin{pmatrix} \langle\zeta\rangle^{n+1} & = & \langle\zeta\rangle^n + \langle\delta\zeta\rangle^{n+1} \\ D^{n+1} & = & h + \langle\zeta\rangle^{n+1} \\ \langle\overline{\mathbf{u}}\rangle^{n+1} & = & \dfrac{\langle(D+\delta\zeta)\overline{\mathbf{u}}\rangle^{n+1}}{D+\langle\delta\zeta\rangle^{n+1}} \end{pmatrix}}_{\text{finalize at}\; n+1}$$

- simple, but equivalent to SM2005
- "fast" terms are derived simply as perturbations. ...in fact, SM2005 is a detour
- no linearization for $\delta\zeta$, but can be: more accurate than linearization for ζ.
  Smallness of $\delta\zeta/D$ is "smaller" than ζ/h

Splitting nonsplittable: non-Boussinesq with compressible EOS

incremental variables

$$\begin{array}{lll} \overline{\rho}D & \to & \overline{\rho}D + \delta m \\ \zeta & \to & \zeta + \delta\zeta \qquad \text{hence} \quad D \to D + \delta\zeta \\ \overline{\rho} & \to & \overline{\rho} + \delta\overline{\rho} \\ \rho_* & \to & \rho_* + \delta\rho_* \end{array}$$

$\Rightarrow$ barotropic continuity

$$\frac{\partial}{\partial t}\delta m + \nabla_\perp\Big((\overline{\rho}D + \delta m)\overline{\mathbf{u}}\Big) = ...$$

identifies δm as the natural fast-time prognostic variable

perturbing non-Boussinesq $\mathscr{F}$

$$\nabla_\perp \left(\frac{\rho_* D^2}{2}\right) - \overline{\rho}D\nabla_\perp h \;\equiv\; \nabla_\perp\left(\frac{\rho_*}{\overline{\rho}}\cdot\frac{\overline{\rho}D\cdot D}{2}\right) - \overline{\rho}D\nabla_\perp h \;\to$$

$$\to\; \nabla_\perp\left[\left(\frac{\rho_*}{\overline{\rho}} + \delta\left(\frac{\rho_*}{\overline{\rho}}\right)\right)\cdot\frac{(\overline{\rho}D + \delta m)\cdot(D + \delta\zeta)}{2}\right] - (\overline{\rho}D + \delta m)\,\nabla_\perp h$$

needs to compute responses $\delta\zeta$ and $\delta(\rho_*/\overline{\rho})$ to changing δm.
Both involve EOS, but are needed in "fast" time $\Rightarrow$ using 2D fields only.

The principal idea: at step n 3D-computed ζ and ρ are in hydrostatic equilibrium as governed by compressible EOS. Once the barotropic mode departs from the state n, the change in δm results in change of both $\delta\zeta$ and $\delta\overline{\rho}$ in such a way that the equilibrium is maintained, hence δm causes proportional changes in $\delta\zeta$ and $\delta\overline{\rho}$,

$$\delta\overline{\rho} = \frac{1}{2}\epsilon\cdot\frac{\delta m}{D}\,, \qquad \delta\zeta = \frac{1-\frac{1}{2}\epsilon}{1+\frac{1}{2}\epsilon\cdot\frac{\delta m}{\overline{\rho}D}}\cdot\frac{\delta m}{\overline{\rho}} \approx \left(1-\frac{1}{2}\epsilon\right)\cdot\frac{\delta m}{\overline{\rho}}$$

where $\epsilon = \frac{gD}{c^2} \ll 1$ is "effective" 2D compressibility; c speed of sound; and quadratically small $\frac{1}{2}\epsilon\cdot\frac{\delta m}{\overline{\rho}D}$ is to keep

$$\overline{\rho}D + \delta m = (\overline{\rho}+\delta\overline{\rho})(D+\delta\zeta)$$

exactly

Assume "factored" (Dukowicz, 2001) form of EOS

$$\rho = r(P)\cdot\left[\rho_0^{\bullet} + \rho^{\bullet\prime}(\Theta, S, P)\right]$$

where $\rho_0^{\bullet} = const$, $\rho^{\bullet\prime} \ll \rho_0^{\bullet}$, and $\left(\partial\rho^{\bullet\prime}/\partial P\right)\Big|_{\Theta,S=const} \ll dr/dP$, $\Rightarrow$ most of pressure effect is absorbed into $r(P)$.

Since $\overline{\rho} = \dfrac{1}{D}\displaystyle\int_{-h}^{\zeta}\rho\,\mathrm{d}z = \dfrac{1}{D}\displaystyle\int_{-h}^{\zeta} r\left(\rho_0^{\bullet} + \rho^{\bullet\prime}\right)\mathrm{d}z = \rho_0^{\bullet}\cdot\dfrac{1}{D}\displaystyle\int_{-h}^{\zeta} r\,\mathrm{d}z + \dfrac{1}{D}\displaystyle\int_{-h}^{\zeta} r\rho^{\bullet\prime}\mathrm{d}z$ introduce

$$\overline{\rho} = \overline{r}\cdot\overline{\rho^{\bullet}} = \overline{r}\left(\rho_0^{\bullet} + \overline{\rho^{\bullet\prime}}\right) \qquad \text{where} \qquad \overline{r} = \frac{1}{D}\int_{-h}^{\zeta} r\,\mathrm{d}z \qquad \text{and} \qquad \overline{\rho^{\bullet\prime}} = \frac{1}{\overline{r}D}\int_{-h}^{\zeta} r\rho^{\bullet\prime}\mathrm{d}z$$

- Change in the total water-column depth causes *proportional stretching* of $\rho^{\bullet}$-profile ($\overline{\rho^{\bullet}}$ is not affected).

- Conversely, $r(P)$-profile moves up-and-down with ζ *without stretching* (as it is a function of pressure only)

$$\delta m = r_{\text{bott}}\rho^{\bullet}_{\text{bott}}\delta\zeta \approx r_{\text{bott}}\rho_0^{\bullet}\delta\zeta$$

  r_{bott} and $\rho^{\bullet}_{\text{bott}}$ are bottom values of r and $\rho^{\bullet}$.

$$\delta m = \left(1 - \frac{1}{2}\epsilon\right)\overline{r}\,\overline{\rho^{\bullet}}\cdot\delta\zeta \approx \left(1 - \frac{1}{2}\epsilon\right)\overline{r}\rho_0^{\bullet}\cdot\delta\zeta$$

  $\Rightarrow$ estimate $\qquad \epsilon = 2\left(1 - \overline{r}/r_{\text{bott}}\right)$

- simplest case: uniform $c \Rightarrow r$ is a linear function of pressure, $\overline{r} = 1 + \epsilon/2$

- more generally: account for the nonuniformity of c due to pressure dependency, **leaving aside** only the temperature effect in the upper ocean

The ratio $(\rho_*/\overline{\rho})$ is affected by *both* compressibility and baroclinicity

$$\rho_* = r_* \rho_*^\bullet = r_* \left(\rho_0^\bullet + \rho_*^{\bullet\prime}\right) \,, \quad \text{where} \quad \begin{cases} r_* = \dfrac{2}{D^2} \displaystyle\int_{-h}^{\zeta} \int_{z}^{\zeta} r \, \mathrm{d}z' \mathrm{d}z \\ \\ \rho_*^{\bullet\prime} = \dfrac{2}{r_* D^2} \displaystyle\int_{-h}^{\zeta} \int_{z}^{\zeta} r \rho^{\bullet\prime} \, \mathrm{d}z' \mathrm{d}z \end{cases}$$

spatially uniform c $\quad \Rightarrow$ estimate $r_* = 1 + \epsilon/3$.

Combining the above,

$$\frac{\rho_*}{\overline{\rho}} = \frac{r_*}{\overline{r}} \cdot \frac{\rho_0^\bullet + \rho_*^{\bullet\prime}}{\rho_0^\bullet + \overline{\rho^{\bullet\prime}}} \approx \left(1 - \frac{1}{6}\epsilon\right) \cdot \left(1 - \frac{\overline{\rho^{\bullet\prime}} - \rho_*^{\bullet\prime}}{\rho_0^\bullet}\right) \,,$$

$\Rightarrow$ an estimate of how $(\rho_*/\overline{\rho})$ responds to perturbations in $\delta\zeta$ and δm,

$$\frac{\rho_*}{\overline{\rho}} = \frac{r_*}{\overline{r}} \cdot \frac{\rho_*^\bullet}{\overline{\rho^\bullet}} \;\rightarrow\; \frac{\rho_*^\bullet}{\overline{\overline{\rho^\bullet}}} \left(1 - \frac{1}{6}\epsilon - \frac{1}{6}\epsilon \cdot \frac{\delta\zeta}{D}\right) \approx \frac{\rho_*^\bullet}{\overline{\overline{\rho^\bullet}}} \left(1 - \frac{1}{6}\epsilon\right) - \frac{1}{6}\epsilon \cdot \frac{\delta m}{\overline{\rho} D}$$

we have neglected quadratically small terms $\mathcal{O}\left(\epsilon^2\right)$ and $\mathcal{O}\left(\epsilon \cdot \dfrac{\overline{\rho^{\bullet\prime}} - \rho_*^{\bullet\prime}}{\rho_0^\bullet}\right)$

- The *baroclinicity ratio* $\rho_*^\bullet/\overline{\rho^\bullet} < 1$ is made of stiffened $\rho_*^\bullet$ and $\overline{\rho^\bullet}$, not full.

Finally, the **fully compressible non-Boussinesq** split

$$\frac{\partial}{\partial t}\Big((\overline{\rho}D+\delta m)\overline{\mathbf{u}}\Big)-\mathscr{F}+gD\nabla_\perp\left[\left(\frac{\rho_*^\bullet}{\overline{\rho^\bullet}}-\frac{1}{2}\epsilon\right)\delta m\right]+g\left(\frac{\rho_*^\bullet}{\overline{\rho^\bullet}}-\frac{1}{2}\epsilon\right)\delta m\nabla_\perp\zeta$$
$$+g\left(\frac{\rho_*^\bullet}{\overline{\rho^\bullet}}-1-\frac{1}{2}\epsilon\right)\delta m\nabla_\perp h$$
$$+g\nabla_\perp\left[\left(\frac{\rho_*^\bullet}{\overline{\rho^\bullet}}-\frac{5}{6}\epsilon\right)\frac{\delta m^2}{\overline{\rho}}-\frac{1}{6}\epsilon\frac{\delta m^3}{\overline{\rho}^2D}\right]=\ldots$$
$$\frac{\partial}{\partial t}\delta m+\nabla_\perp\Big((\overline{\rho}D+\delta m)\overline{\mathbf{u}}\Big)=\ldots$$

p_b evolves in slow time by adding increments $g\cdot\delta m$ computed (fast-time averaged) by barotropic mode.

Before running barotropic mode from baroclinic step n to $n+1$ all necessary terms – full vertically integrated PGF $\mathscr{F}$; free-surface ζ, stiffened baroclinic ratio $\rho_*^\bullet/\overline{\rho^\bullet}$; effective 2D compressibility ϵ – are precomputed using full 3D algoritms with compressible EOS. They are kept constant thereafter until the next baroclinic time step.

Then

$$\begin{pmatrix}\delta m=0\\ \\ \overline{\mathbf{u}}=\langle\overline{\mathbf{u}}\rangle^n\end{pmatrix}\rightarrow\begin{pmatrix}\langle\delta m\rangle^{n+1}\\ \langle(\overline{\rho}D+\delta m)\overline{\mathbf{u}}\rangle^{n+1}\\ \langle\langle(\overline{\rho}D+\delta m)\overline{\mathbf{u}}\rangle\rangle^{n+1/2}\end{pmatrix}\rightarrow\begin{pmatrix}\langle p_b\rangle^{n+1} & = & \langle p_b\rangle^n+g\langle\delta m\rangle^{n+1}\\ \\ \langle\overline{\mathbf{u}}\rangle^{n+1} & = & \dfrac{\langle(\overline{\rho}D+\delta m)\overline{\mathbf{u}}\rangle^{n+1}}{\overline{\rho}D+\langle\delta m\rangle^{n+1}}\end{pmatrix}$$

- Note $\left(\frac{\rho_*^\bullet}{\overline{\rho^\bullet}}-\frac{1}{2}\epsilon\right)$ is **not equal to** non-stiffened $\frac{\rho_*}{\overline{\rho}}$

Summary for non-Boussinesq splitting

- complex, but possible
- "fast" terms capable to capture correct barotropic phase speed accurately accounting for both
  1. slow-down due to stratification (relatively to uniform density)
  2. slow-down due to compressibility (relatively to incompressible fluid of the same depth)
- Splitting accuracy comparable to Boussinesq ROMS, *second-order* with respect to relevant small parameters

$$\frac{\rho^{\bullet\prime}}{\rho_0^{\bullet}} \approx 0.005 - 0.01 \qquad \epsilon = \frac{gh}{c^2} \precsim 0.025$$

$$\frac{\Delta\left(c^2\right)}{c^2} \approx 0.04 \qquad \frac{\zeta}{h} \approx 2 \times 10^{-4} - 0.2$$

Conclusion

- Boussinesq approximation is still useful

- EOS stiffening removes internal contradictions within the Boussinesq model

- Most complexity associated with non-Boussinesq mode splitting are due to bulk compressibility terms. These are believed not to lead to physically important/interesting phenomena

- Splittings are derived via perturbations