Practical Simulation of Surface Tension Flows

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1 Introduction

Visual effects often requires the simulation of natural phenomenon such as water, explosions, steam, or fire. For *The Cat in the Hat*, Rhythm and Hues was required to simulate a character-animated fish interacting with water in a fishbowl.

In small-scale fluid motion, surface tension provides an important visual cue for depicting the physical size of the underlying fluid. Waves driven by surface tension, called capillary waves, travel much faster over the water surface than waves driven by gravity, which are generated at larger scales.

In addition to small-scale waves, surface tension forces also drive larger-scale gross motion of the fluid. This can be seen in the fishbowl, shown in Figure 1. Without surface tension, the surface waves would reflect until numerical viscosity damped them out, and the fish bowl would look more like a swimming pool. To simulate such a small-scale water surface required developing new techniques for modeling surface tension.

Numerically, simulating surface tension is difficult because it inherently imposes a severe time-step restriction due to the propagation speed of capillary waves. The time step restriction due to capillary wave propagation is $O(\Delta x^{1.5})$, while the standard CFL time step restriction is only $O(\Delta x)$. In conventional methods for solving the Navier-Stokes equations, the total simulation time is quickly dominated by surface tension as higher resolutions are used. This is one reason computer graphics practitioners have tended to ignored it.

We have developed a split time scale method ([Yanenko 1971]) that separates the fast-changing surface tension term from the slowly-changing terms bound by a less restrictive CFL-time step condition. This new scheme reduces the number of computationally expensive steps, while accurately calculating fast wave propagation within the framework of existing fluid simulators.

2 Numerical Method

Using finite time step Δt, a fluid system may be updated using the rule,

$$u^{t+\Delta t} = u^t + \Delta t \left[F(S^t) + g - (u^t \cdot \nabla) u^t - \nabla p^t / \rho^{t+\Delta t} \right]$$

where $u^t(x)$ is the time-dependent velocity field describing the fluid flow at point x at time t, $p^t(x)$ is the fluid pressure, S^t is the surface geometry, $F(S^t)$ is the surface tension force, g is the gravity vector, and $\rho^t(x)$ is the fluid density. We must also update the surface geometry,

$$S^{t+\Delta t} = advect(S^t, u^t, \Delta t)$$

where *advect* is a procedure (such as the particle-level set method [Enright et al. 2002]) that evolves the shape of the air/water interface under the influence of the flow field u^t over the timespan from t to $t + \Delta t$.

Our technique splits the update equations into two parts, one containing only the surface tension term and the other containing all terms. The surface tension terms are "cycled" at the rate determined by capillary wave propagation, while the overall equation is only updated at the rate determined by the CFL condition. This gives rise to the following multiple-time scale update rule:

$$\begin{aligned}
u_*^{t+\frac{1}{n}\Delta t} &= u^t + \frac{\Delta t}{n} \left[F(S^t) \right] \\
&\dots \\
u_*^{t+\frac{(n-1)}{n}\Delta t} &= u_*^{t+\frac{(n-2)}{n}\Delta t} + \frac{\Delta t}{n} \left[F(S^{t+\frac{(n-2)}{n}\Delta t} \right] \\
u^{t+\Delta t} &= u^t + \Delta t \left[F(S^{t+t+\frac{(n-1)}{n}\Delta t}) + g - (u^t \cdot \nabla) u^t - \nabla p^t / \rho^{t+\Delta t} \right]
\end{aligned}$$

At substep i, the surface geometry is advanced using

$$S^{t+\frac{i}{n}\Delta t} = advect(S^{t+\frac{i-1}{n}\Delta}, u_*^{t+\frac{i-1}{n}\Delta}, \frac{\Delta t}{n}).$$

Because the full equation is updated less frequently than just the surface tension terms, there is a significant savings in the number of times the pressure must be calculated, which requires solving a large sparse linear system. For medium resolution simulations, we typically see a speedup of a factor of 2 over the naive single time scale method. This is the difference between an unacceptable long 30 hours, and a 15 hour simulation which can run overnight.

Figure 1: Frame from *Cat in the Hat* including surface tension effects. *Cat in the Hat* ™©2003 Universal/DreamWorks.

References

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