Phase matrix and cross sections for single scattering by circular cylinders: a comparison of ray optics and wave theory

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The phase matrix and several quantities for single scattering by an arbitrarily oriented circular cylinder are formulated by using the approximation of ray optics, which includes geometrical reflection and refraction plus Fraunhofer diffraction; then the effects of polarization are considered. Computations were made using electromagnetic wave theory and ray optics approximations for \( m = 1.31 - 0.0i \) and \( 1.31 - 0.1i \). Results by these methods approach one another as the ratio of the cylinder's circumference to the incident wavelength increases. One of two ray optics approximations proposed requires less computation time than wave theory. The applicability of the ray optics approximation is dependent on the orientation of the cylinder relative to the incident light as well as the size parameter and, moreover, dependent on what quantity for single scattering is compared.

I. Introduction

The average length of ice crystals in cirriform clouds is several hundred micrometers and considerably larger than the wavelength of radiation in the visible and near-IR regions, so that the ray optics approximation seems to be suitable. In order to obtain information on the light scattering properties of columnar ice crystals of hexagonal cylindrical shape, we shall consider in this study the light scattering by infinitely long homogeneous isotropic cylinders. The solution of scattering of electromagnetic plane waves by an infinitely circular cylinder was derived by Rayleigh at normal incidence and by Wait at oblique incidence. Since then numerous calculations have been performed by many investigators. Recently the method of integral equations was applied to a hollow hexagonal cylinder. To determine the diameter and the refractive index of a fiber, some approximate computations of backscattered light intensities from a circular cylinder have been performed by means of geometrical optics. Similar computations of backscattered light intensities have been done by Holoubek and Saekeang and Chu in a more elaborate fashion. But a ray optics formulation including effects of polarization for light scattering by an arbitrarily oriented cylinder has not yet been performed. We shall develop the formulas of light scattering by a cylinder on the basis of the ray optics approximation and compare computation results by this approximation with those by the exact solution of wave theory. The purpose of this comparison is to verify applicability of the ray optics approximation and to find the range of validity.

II. Scattering Geometry

Scattering geometry is shown in Fig. 1. In this study, we follow the geometry and notation by Liou. The \( Z \) axis of the cylindrical coordinate system \((r, \phi, Z)\) is placed along the cylinder axis so that the cylinder occupies the region \( r \leq a \), where \( a \) is the radius. The \( X \) axis is in the plane containing the direction of the incident light and the \( Z \) axis. The complement of the angle between the incident direction and the \( Z \) axis is denoted as \( \beta \). The angle between the normal of the cylinder at the incident point, i.e., the radial direction of the cylinder and the negative \( X \) axis, is specified as \( \phi \). Usually, we have only to consider rays with \( 0 \leq \phi \leq \pi/2 \). Scattered radiation emerges along the surface of the cone with an apical angle \( \tau - 2\alpha \), so that it can be expressed by \( \phi \) denoted as an observation angle. We consider two polarization states of light, where electric vector \( E \) vibrates parallel to the plane containing the direction of propagation of light and the \( Z \) axis [case (0)] and the electric vector vibrates perpendicular to this plane [case (r)].

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Fig. 1. Geometry for light scattered by an infinitely long circular cylinder. All symbols are explained in the text.

III. Ray Optics

We denote an amplitude matrix by \( T \) and a phase matrix by \( G \), which transform the amplitude of electromagnetic fields and the Stokes parameters \( (I, I_r, U, V) \), respectively, of the incident light to those of the scattered light in a reference plane containing the direction of propagation of light and the \( Z \) axis. Amplitude matrix \( T \) thus defined is given by

\[
\begin{align*}
T & = \left( \begin{array}{cccc}
\hat{E}_i & \hat{E}_1 & \hat{E}_2 & \hat{E}_3 \\
\end{array} \right) \\
& = \left( \begin{array}{cccc}
\frac{2}{\pi k R \cos \alpha} \exp[-i(k R + \omega t)] & 0 & 0 & 0 \\
-3\pi/4 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{array} \right) \\
& \left( \begin{array}{cccc}
\hat{E}_i & \hat{E}_1 & \hat{E}_2 & \hat{E}_3 \\
\end{array} \right),
\end{align*}
\]

where \( \hat{E}_i \) and \( \hat{E}_s \) are, respectively, the incident and scattered amplitudes of the electric vector, \( k \) is the wave number in vacuum, \( \omega \) is the circular frequency, \( R \) is the distance from the apex of the cone, \( t \) is time, and \( i = (-1)^{1/2} \). We alternatively define a matrix \( A \) as a multiple of \( T \):

\[
A_j(x, m, \alpha, \phi, \alpha, \beta) = \left( \begin{array}{cccc}
\frac{2}{\pi k R \cos \alpha} \exp[-i(k R + \omega t)] & 0 & 0 & 0 \\
-3\pi/4 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{array} \right) \\
\left( \begin{array}{cccc}
\hat{E}_i & \hat{E}_1 & \hat{E}_2 & \hat{E}_3 \\
\end{array} \right),
\]

where \( x = (2\pi a/\lambda) \) denotes that the ratio of the circumference of the cylinder to the wavelength of incident light, \( m = m_r - im_i \) is the complex index of refraction of the scattering medium. The matrix \( A \) is the amplitude matrix normalized by the square root of a multiple of the geometric shadow area \( 2a \cos \alpha \) per unit length of the cylinder, and its elements are expressed in the form

\[
A_j(x, m, \alpha, \phi, \alpha, \beta) = \left( \begin{array}{cccc}
\delta_{ij} & \delta_{ij} & \delta_{ij} & \delta_{ij} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array} \right),
\]

where \( \delta_{kl} \) denotes the Kronecker delta, i.e., \( \delta_{kl} = 1 \) for \( k = l \) and zero otherwise. \( A_p \) is the contribution from Fraunhofer diffraction that can be expressed from the scalar Kirchhoff approximation as

\[
A_p(x, \alpha, \phi) = (2x \cos \alpha)^{1/2} \frac{\sin(x \cos \alpha \sin \phi)}{x \cos \alpha \sin \phi}.
\]

The derivation of this formula is described in Appendix.

A. Since the factor \( x \cos \alpha \) is included as a single parameter in Eq. (4), it is expected that the cylinder apparently shrinks as \( \alpha \) becomes large. \( A_j^{(p)} \) is the contribution from geometrical optics, which includes external reflection \( (p = 0) \), refraction \( (p = 1) \), and internal reflection of each order \( (p \geq 2) \); hereafter we call them the transmitted light of the \( p \)th order. The summation for \( t \) from \( 1 \) to \( M_p \) represents the contribution from rays with different incident points \( \beta \) and the same \( p \). Clearly \( M_p \) is unity for \( p = 0 \) and for \( p = 1 \), but for \( p \geq 2 \), \( M_p \) can be more than 2 since \( \phi(\alpha, \beta) \) is a multivalued function of \( \beta \) (Ref. 5, p. 229, and Ref. 13). The value \( N \) should theoretically be infinity, but in practice a small integer can be adopted with sufficient accuracy; \( N = 4 \sim 9 \) are sufficient to account for more than 99.9% of the scattered energy even for nonabsorbing cylinders except for cases with very large values of \( \alpha \). Amplitude matrix elements \( A_j^{(p)} \) are given by

\[
A_j^{(p)}[x, m, \alpha, \phi(\alpha, \beta)] = c_{j(p)} \left( \frac{\pi \cos \beta}{|\delta_{p}/\delta_{p}|} \right)^{1/2} \exp(i\gamma),
\]

where

\[
c_{j(p)} = \begin{cases}
L_{x1} RL_{x1}, & \text{for } p = 0, \\
L_{x1} T (L_{x2} CR)^{p-1} L_{x2} TL_{x1}, & \text{for } p \geq 1.
\end{cases}
\]

Matrix \( L_\delta \) denotes the rotation of the coordinate system:

\[
L_\delta = \begin{pmatrix}
\cos \delta & \sin \delta \\
-\sin \delta & \cos \delta
\end{pmatrix},
\]

\( R \) is the reflection matrix, and \( T \) is the transmission matrix:

\[
R = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
\]

\[
T = \begin{pmatrix}
1 - r_{\parallel}^2 & 0 \\
0 & (1 - r_{\perp}^2)^{1/2}
\end{pmatrix}
\]

where \( r_{\parallel} \) and \( r_{\perp} \) are the Fresnel reflection coefficients parallel and perpendicular, respectively, to the incident plane, i.e., the plane containing the directions of propagation of incident and reflected rays. The Fresnel reflection coefficients are given by

\[
r_{\parallel} = \cos \theta_{i} - \cos r_{\parallel},
\]

\[
r_{\perp} = \cos \theta_{i} - \cos r_{\perp}.
\]

In the above equations, \( \tau_i \) denotes the incident angle with reference to the normal,

\[
\cos \tau_i = \cos \alpha \cos \beta,
\]

and \( \tau_r \) denotes the refracted angle given by Snell's law:

\[
m \sin \tau_r = \sin \tau_i.
\]

To determine the refracted direction, \( \tau_r \) is approximated with the angle obtained by replacement of \( m \) with \( m_r \) in Eq. (11). In Eq. (6), matrix \( C \) is given by
This matrix expresses a phase shift of $\pi$ due to internal reflection. The relations $A_{1}^{(p)} = -A_{3}^{(p)}$ and therefore $A_{4} = -A_{3}$ are valid from the property of $e^{ip\tau}$. Since the incident angle of any ray refracted once into the cylinder is always $\tau$, the rotation angles of the coordinate system $\delta 1$ and $\delta 2$ are obtained from the spherical geometry shown in Figs. 2(a) and (b):

$\sin \delta 1 = \sin \beta / \sin \tau$,  
$\cos \delta 2 = -\cos \Delta + \sin \Delta \cos (2\xi)$,  

where $\Delta$ and $\xi$ are auxiliary angles:

$\sin \Delta = \sin \alpha / \sin \tau$,  
$\tan \xi = \cos \Delta \tan \tau$.  

From Fig. 2, it can also be understood that the scattered light is confined to the surface of a solid cone that forms an angle $\alpha$ with the X-Y plane. The factor $(\pi \cos \beta) / (\partial \phi / \partial \beta)^{1/2}$ in Eq. (5) is derived from the conservation of the flux of radiant energy. Deviation angle $\phi'$ appearing in this factor is given by

$\phi'(\alpha, \beta) = \pi - 2\beta - p(\pi - 2\xi)$.  

This angle may express how many times and in what direction the ray rotates within the cylinder. The derivative $\partial \phi'/\partial \beta$ can be obtained straightforwardly from Eq. (17).

$\phi'(\alpha, \beta) = 2\pi k + q \phi(\alpha, \beta)$,  

where $k$ is an integer, and $q$ is $+1$ or $-1$. Namely, $\phi$ is the projection of $\phi'$ on the branch $[0, \pi]$ with a proper choice of $k$ and $q$. The factor $\gamma$ in Eq. (5) is a complex phase of a light ray except for that of the Fresnel coefficient already included in $e^{ip\tau}$:

$\gamma = \frac{3}{4} \pi + \delta + \left[ \rho + \frac{1}{2} (s - 1) \right] \frac{1}{2} \pi$,  

where

$\delta = 2\pi (\cos \rho - \rho \cos \tau) + \frac{2\pi m}{\lambda} \rho$,  

$l = \frac{2\pi \cos \tau}{1 - \sin^2 \alpha / m^2}$,  

$s = \text{sgn} \left( \frac{\partial \phi}{\partial \beta} - \frac{\partial \phi'}{\partial \beta} \right)$.  

The constant phase factor in Eq. (19) is taken to be $3\pi / 4$, whereas this factor is $\pi / 2$ in the case of a sphere (Ref. 5, p. 207). The real part of $\delta$ is the advance in phase of the actual ray due to the length of optical path with respect to a hypothetical ray scattered without phase lag at the apex of the cone. This term is exactly the same as the corresponding quantity for a sphere (Ref. 5, p. 207). The imaginary part of $\delta$ accounts for absorption within the cylinder, where $\rho l$ is the total path length in the cylinder. The quantity $p + (s - 1) / 2$ in Eq. (19) is the number of focal lines through which the ray passes. With respect to the phase factor, it is proposed that the term $[-2k + (1 - q) / 2]{\pi / 2}$, which appears in the case of a sphere (Ref. 5, p. 207), should not be included in the case of a cylinder because of the absence of the glory for nonspherical particles.

Now let us consider a cylinder at an oblique incident angle $\alpha$, say, cylinder $a$, which scatters a light ray to an observation angle $\beta$. Let us also take another cylinder, $c$, in the mirror image of cylinder $a$ with respect to the scattering plane, which necessarily has the same oblique incident angle $\alpha$ and scatters a ray in the same direction in space as does cylinder $a$. The cylinder at the reciprocal position of cylinder $a$, say, cylinder $b$, and its mirror image, say, cylinder $d$, coincide with cylinders $a$ and $c$, respectively. Taking these symmetry relations into consideration, the 4 X 4 phase matrix of these four cylinders, which transforms the Stokes parameters $(I, I, U, V)$ from incident to scattered, is given, in van de Hulst's notation (Ref. 5, p. 44), by

$$G = \begin{pmatrix} M_2 & \frac{1}{2} (M_3 + M_4) & 0 & 0 \\ \frac{1}{2} (M_3 + M_4) & M_1 & 0 & 0 \\ 0 & 0 & S_{12} + S_{34} & -D_{21} \\ 0 & 0 & D_{21} & S_{12} - S_{34} \end{pmatrix}.  \tag{23}$$

Fig. 2. (a) Spherical geometry of the externally reflected light. $O$ denotes the incident point. $(i), (n_0),$ and $(r_0)$ express the incident ray, the normal, and the externally reflected ray, respectively. (b) Spherical geometry of the transmitted light of the first order. $(r_1), (n_1),$ and $(i_1)$ express the refracted ray, the normal, and the transmitted ray of the first order, respectively.
where we consider a direction for which scattering angle \( \theta \) is neither zero nor \( \pi \). From the relation \( A_4 = -A_5 \), we can easily find the relations \( S_{34} = -M_3 \) and \( M_4 = M_3 \). Then we have

\[
G = \begin{pmatrix}
M_2 & M_3 & 0 & 0 \\
M_3 & M_1 & 0 & 0 \\
0 & 0 & S_{12} - M_5 & -D_{21} \\
0 & 0 & D_{21} & S_{12} + M_5
\end{pmatrix}.
\]

In Eq. (24), five independent parameters occur for a polydispersion, and the relation

\[
G_{44} - G_{33} = 2G_{12} (\neq 0)
\]

is easily found. After \( \phi' \) intersects \( n \pi \) odd times \( (n = \text{an integer}) \), the ray with \( -\pi/2 \leq \beta \leq 0 \) contributes to the amplitude, so that the signs of \( \delta_1 \) and \( \delta_2 \) change. Therefore, whenever \( \phi'(\alpha, \beta) \) takes the value of \( n \pi \), \( A_4^{(p)} \) and \( A_3^{(p)} \) should be multiplied further by \( -1 \). In such a case, at \( \phi = 0 \) and \( \phi = \pi \), there are two contributions to the amplitude coming from each side of a cylinder whose magnitudes are the same but with opposite signs, so that \( G_{12} \) should always be zero at these directions. On the other hand, \( G_{11} \) and \( G_{22} \) can have finite and small peaks at \( \phi = 0 \) or \( \phi = \pi \) because there are two contributions to the amplitude coming from each side of a cylinder of the same magnitude and sign. A typical example is the case for \( \alpha = 45^\circ \) shown later.

For the incidence of unpolarized natural light with Stokes parameters \((\frac{1}{2}, \frac{1}{2}, 0, 0)\), the phase function and the degree of linear polarization are given, respectively, by

\[
G = \frac{1}{2} (G_{11} + G_{12} + G_{21} + G_{22})\),
\]

\[
DP = \frac{(G_{11} + G_{12}) - (G_{21} + G_{22})}{(G_{11} + G_{12}) + (G_{21} + G_{22})}.
\]

If the phase matrix of randomly oriented cylinders is needed, the scattering plane should be taken as the reference plane. However, phase function \( G \) and element \( G_{44} \) are independent of the choice of the reference plane.

The scattering efficiency factor is defined by the total scattered energy integrated over all angles \( \phi \) divided by the product of the incident intensity and the geometric shadow area per unit length. In the case of ray optics, with the definition of \( A_j \) in Eq. (2), this is given by

\[
Q_{\text{sc}} = \frac{1}{2} (Q_{\text{sc},A} + Q_{\text{sc},B})
\]

\[
= \frac{1}{2} \left( \int_0^{2\pi} (G_{11} + G_{12}) d\phi + \int_0^{2\pi} (G_{21} + G_{22}) d\phi \right)
\]

\[
= \frac{1}{\pi} \int_0^{2\pi} G d\phi.
\]

And, accordingly, the scattering cross section per unit length of a cylinder is given as

\[
C_{\text{sc}} = 2\pi \cos \alpha Q_{\text{sc}}.
\]

The absorption efficiency factor \( Q_{\text{abs}} \) is the difference between the total energy impinging on the cylinder, which is assumed to be unity, and the externally reflected energy \( E^{(0)} \) plus the transmitted energy \( E^{(p)} (p \geq 1) \); i.e.,

\[
Q_{\text{abs}}(x, m, \alpha) = \frac{1}{2} (Q_{\text{abs},A} + Q_{\text{abs},B})
\]

\[
= \frac{1}{2} \left[ \left( 1 - 2 \sum_{p=0}^{\infty} [E^{(p)} + E^{(p)}] \right) 
\]

\[
= 1 - \sum_{p=0}^{\infty} E^{(p)},
\]

where

\[
E^{(p)} = \frac{1}{2} \int_0^{2\pi} \left| (e^{(p)}) \right|^2 \cos \beta d\beta.
\]

The extinction efficiency factor can be obtained as

\[
Q_{\text{ext}} = Q_{\text{sc}} + Q_{\text{abs}}.
\]

When many cylinders with an oblique incident angle \( \alpha \) are randomly located, the intensities from the individual cylinders may be added. Therefore, phase matrix elements \( G_{kl} \) for a polydispersion of cylinders are given by the following integration:

\[
G_{kl}(x, m, \alpha, \phi) = \int G_{kl}(x, m, \alpha, \phi) n(x) dx = \int G_{kl}(x, m, \alpha, \phi) n(x) dx,
\]

where \( n(x) dx \) is the number of particles per unit volume with size parameter between \( x \) and \( x + dx \). It is noted that if \( T \) is used as the amplitude matrix instead of \( A \), the size distribution \( n(x) \) must be substituted instead of \( n(x) \) in Eq. (33).

The method described so far to obtain phase matrix \( G \), adding the complex amplitudes and taking the squared modulus of the combined amplitude, which will be called ray optics A, is exact in the framework of ray optics, but this method still requires a fairly long computation time. On the other hand, if a phase matrix for a polydisperse system is needed, it will be a good approximation to superpose the intensities of the diffracted light and the transmitted light of each order. This alternative method of ray optics will be called ray optics B. For the latter case, the phase matrix elements are expressed as

\[
G_{kl}(x, m, \alpha, \phi) = G_{kl}^{(0)}(x, m, \alpha, \phi)
\]

\[
= \sum_{p=0}^N M_p \sum_{t=1}^N G^{(p)}_{kl} [x, m, \alpha, \phi(\alpha, \beta_t)],
\]

where \( G_{kl}^{(0)} \) and \( G_{kl}^{(p)} \) are phase matrix elements generated, respectively, from \((\delta_{1j} + \delta_{2j}) A^D \) and \( A^{(p)} \) by Eq. (23). The elements \( G_{kl}^{(p)} \) can be obtained by \( \delta_{kl} |A|^2 \).

In ray optics B, the element \( G_{12} \) is taken to be zero at \( \phi = 0 \) and \( \phi = \pi \), similarly to ray optics A. If there is a contribution to the intensity at \( \phi = 0 \) or \( \pi \) from a ray whose \( \beta \) is not zero, the elements \( G_{12}^{(0)}, G_{22}^{(0)}, G_{12}^{(p)} \), and \( G_{12}^{(p)} - G_{12}^{(0)} \) of the corresponding \( t \) should be multiplied...
The absorption efficiency factor is obviously given by Eq. (30).

To consider the ground for ray optics B, let us rewrite Eq. (3) as

\[ A_j = \sum_{s=0}^{K} a_j^{(s)} \exp[i(\delta_j^{(s)} + \sigma_j^{(s)})], \quad j = 1, 2, 3, 4, \tag{36} \]

where \( \delta_j^{(s)} \) is the phase factor depending on polarization component \( j \), \( \sigma_j^{(s)} \) is the phase factor common to all polarization components, \( a_j^{(s)} \) is the real amplitude of the light ray, \( K \) is the number of light rays contributing to the intensity at a direction \( \phi \). For example, \( \delta_j^{(s)} = [\text{the phase of } \epsilon_j^{(s)}] \), and \( \sigma_j^{(s)} = \gamma \) for the transmitted light of each order; \( \delta_j^{(s)} = \sigma_j^{(s)} = 0 \) for the diffracted light. Let us consider the phase matrix element \( D_{21} \) in Eq. (24) as an example. Using Eq. (36), \( D_{21} \) can be expressed as

\[ D_{21} = \sum_{s=0}^{K} a_2^{(s)} a_1^{(s)} \sin[\delta_1^{(s)} - \delta_2^{(s)}] \]

\[ + \sum_{r=0}^{K} \sum_{u=0}^{K} a_1^{(s)} a_2^{(u)} \sin[\delta_1^{(s)} - \delta_2^{(u)}] \]

\[ - \delta_2^{(s)} + \sigma_2^{(s)} - \sigma_1^{(s)}. \tag{37} \]

If this quantity is integrated over a wide range of size parameters, the double summation of Eq. (37) will be nearly zero, since \( \sigma_2^{(s)} - \sigma_1^{(s)} \) varies over a wide range. This means that the phase matrix element is composed of superposition of the intensities of the diffracted light and the transmitted light of each order. Moreover, one can easily see that if the scattering body is isotropic and nonabsorbing and no total internal reflection occurs, the element \( D_{21} \) becomes nearly zero for a polydisperse system, since \( \delta_1^{(s)} - \delta_2^{(s)} \) in Eq. (37) is \( \pm \pi \) or zero from the fact that \( \delta_0^{(s)} = [\text{the phase of } \epsilon_0^{(s)}] \) or zero.

\[ Q_{\text{ext}} = \frac{1}{2} \left( Q_{\text{ext},1} + Q_{\text{ext},r} \right) \]

\[ Q_{\text{sca}} = \frac{1}{2} \left( Q_{\text{sca},1} + Q_{\text{sca},r} \right) \]

The absorption efficiency factor is obviously given by Eq. (30).

V. Numerical Results and Discussion

In geometrical optics, to obtain \( A^{(p)} \) as a function of \( \phi \) it is necessary to solve a 2pth-order algebraic equation for the pth-order transmitted light, which is the reverse solution of Eq. (17), i.e., \( \beta = \beta(\alpha, \phi) \). Computations using geometrical optics were carried out for 181 equally spaced incident directions with angular increments of 5°. The real amplitude and the phase of \( A^{(p)} \) for ray optics A and \( G^{(p)} \) for ray optics B were linearly interpolated at 199 observation angles, 0.0(1)2(1)180°. But except for Fig. 3, figures are drawn at intervals of 2° in \( \phi \) for the sake of comparison with exact wave theory. The computational instability due to the rainbow was removed by the method described in Appendix B. In both cases of ray optics and wave theory, the normalized phase function \( G/Q_{\text{sca}} \) is drawn in the figures, and, correspondingly, \( G_{\theta}^2 \)’s generated from amplitude matrix \( A \) are replaced by \( G_{\theta}^2/Q_{\text{sca}} \), where

\[ T_4(x,m,\alpha,\phi) = \sum_{n=1}^{\infty} a_n \sin \phi. \tag{41} \]

In the above equations, the coefficients \( a_1, a_2, b_1, \) and \( b_2 \) are determined by the boundary condition of continuity of the tangential components of the electromagnetic field at the interface. These coefficients are functions of the Bessel functions of the first kind, the Hankel functions of the second kind, and their first derivatives, whose arguments involve \( x, m, \) and \( \alpha \). The detailed expressions for these are found, e.g., in Ref. 7. It is noted here that the relations \( b_1 = -a_1 \) and therefore \( T_4 = -T_3 \) are always valid and that \( T_3 = T_4 = 0 \) at \( \phi = 0 \) and \( \phi = \pi \).

The phase matrix is generated from \( T \) by means of Eq. (23). The efficiency factors for extinction and scattering are given, respectively, by

\[ Q_{\text{ext}} = \frac{1}{2} \left( Q_{\text{ext},1} + Q_{\text{ext},r} \right) \]

\[ = \frac{2}{2 \cos \alpha} \left[ \sum_{n=1}^{\infty} \left( \Re[a_0] + 2 \sum_{n=1}^{\infty} \Re[b_n] \right) \right] \]

\[ + \left( \Re[a_0] + 2 \sum_{n=1}^{\infty} \Re[a_n] \right) \]

\[ Q_{\text{sca}} = \frac{1}{2} \left( Q_{\text{sca},1} + Q_{\text{sca},r} \right) \]

\[ = \frac{2}{2 \cos \alpha} \left[ \left| b_0 \right|^2 + 2 \sum_{n=1}^{\infty} \left| b_n \right|^2 \right] \]

\[ + \left( \left| a_0 \right|^2 + 2 \sum_{n=1}^{\infty} \left| a_n \right|^2 \right) \].
Fig. 3. Phase functions for a single cylinder at normal incidence. The uppermost, middle, and lowermost curves are those computed from ray optics B, ray optics A, and wave theory, respectively. The vertical scale applies to the lowermost curve. The other curves are successively displaced upward by factors of 10, with the horizontal bars occurring where the phase function has the value of unity.

\[ Q_{\text{ext}} = \int Q_{\text{scat}} n(x) dx / \int x n(x) dx. \]  

(44)

As the complex refractive index of ice, the value of 1.31–0.0i at a wavelength of 0.7 \( \mu m \) was adopted.

Figure 3 illustrates the phase functions for a cylinder with 10-\( \mu m \) radius at normal incidence (\( \alpha = 0^\circ \)). The uppermost curve is that computed from ray optics B by adding the intensities. The steep feature at the very forward direction results from the diffraction. The broad forward lobe results from the transmitted light of the first order, and the primary and secondary rainbows appear concurrently at the observation angle of 135° for \( m = 1.31–0.0i \). The middle curve is that computed from ray optics A by adding the complex amplitudes so that the effect of interference is seen overlapping the feature of the uppermost curve. The lowermost curve is that computed from the exact wave theory. Although the frequency of fluctuation due to interference is higher in this case, the overall tendency coincides well with that of the middle curve.

For a large size parameter, since the angular scattering pattern varies rapidly, a comparison should be made with a sufficiently high resolution. However, as the ice crystals in real clouds are not monodisperse but polydisperse, it is reasonable to integrate the contributions over a size range that the fluctuations due to interference are averaged out. Thus we can accomplish meaningful comparisons with the resolution mentioned above. For the particle size distribution we employed the modified gamma function given by

\[ n(x) \propto x^b \exp(-bx), \]  

(45)

where \( b \) is a constant determined by the average size. Sufficiently wide range was adopted, and the range of integration was divided into sufficiently small intervals in Eq. (33).

Figure 4 compares phase functions at normal incidence for ray optics and wave theory. The average radii are 5, 10, and 30 \( \mu m \), and corresponding size parameters are 44.9, 89.8, and 269.3, respectively. The phase functions for each size are successively displaced by 1 order of magnitude to allow a clear comparison. We can see that the glory at \( \phi = 180^\circ \) that appears for...
spherical particles\textsuperscript{16,17} does not occur in this case and that the normalized intensity of diffracted light at $\phi = 0^\circ$ is proportional to $x$ in this case [Eq. (4)] instead of $x^2$ for spheres. Some explanations for scattering properties of spherical particles given by Liou and Hansen\textsuperscript{16} can be applied to those of cylinders computed by ray optics B and wave theory shown in Fig. 4 and following figures.

In Fig. 5 are compared phase functions obtained by ray optics and wave theory for the oblique incident angles $\alpha = 45^\circ$. The primary rainbow, the secondary rainbow, and the rainbow due to the $p = 5$ component occur, respectively, at observation angles of $\sim 163$, $84$, and $128^\circ$. The rainbow due to the $p = 4$ component at $\phi \approx 24^\circ$ is not observed clearly because the intense transmitted light of the first order obscures it. The phase function computed by ray optics B does not vary with $\bar{x}$ in the range of backward observation angles ($\phi \geq 90^\circ$) where no diffracted light contributes. This is also the case for other matrix elements produced by ray optics B. On the other hand, as seen from Fig. 5, the phase functions computed by ray optics A vary with $\bar{x}$ ($= 2\pi\bar{a}/\lambda$) even in the range of backward observation angles. Such variations in the case of finite $\bar{x}$ are caused by the effect of interference or the term $\gamma$ in Eq. (5). It is expected that ray optics B is coincident with ray optics A in the limit of infinitely large $\bar{x}$. Indeed, we can see in the vicinity of the primary rainbow in Fig. 5 that ray optics A and wave theory, which are close to each other, approach ray optics B with an increase of $\bar{x}$.

Figure 6 shows comparisons of phase functions by ray optics and wave theory for $\alpha = 85^\circ$. Since calculations by ray optics A were performed at an early stage of this study, only rays up to $p = 9$ were included, but rays up to $p = 12$ were included by ray optics B. Therefore additional rainbows due to components with $p = 10$ and $12$ appear at $\phi \approx 180^\circ$ and $\phi \approx 170^\circ$, respectively, in ray optics B. For small oblique incident angles $\alpha$, the phase functions have anisotropic forwardly intense features due to contributions from the diffracted light and the transmitted light of the first order, while for large values of $\alpha$, especially $\alpha \gtrsim 60^\circ$, they have relatively isotropic features due to the contribution from the externally reflected light.

In Fig. 7, we compare the cross-polarization element $G_{12}$ ($= G_{21}$) computed from wave theory with those computed from ray optics at the oblique incident angles $\alpha = 45^\circ$ and $85^\circ$. The curves for $\alpha = 85^\circ$ are displaced upward by factors of $10^2$. For $\alpha = 0^\circ$, this element is always zero, because the incident plane is always in the circular cross section of the cylinder. The energy contained within the element $G_{12}$ increases monotonically as the oblique incident angle $\alpha$ increases. As mentioned before, the intensities of the cross-polarization element are zero at $\phi = 0^\circ$ and $180^\circ$. For $\alpha = 85^\circ$, we can see a subpeak at $\phi \approx 4^\circ$ for both wave theory and ray optics B, which is attributable to the rainbow due to the $p = 11$ component. For $\alpha = 45^\circ$, in spite of the discontinuous damping of this element at $\phi = 180^\circ$, the phase function computed by ray optics B is coincident with wave theory, which are close to each other, approach ray optics B with an increase of $\bar{x}$.
Thus, at the observation angle $\phi \approx 90^\circ$, the cross-polarization element $G_{12}$ has a maximum value, and the polarization elements $G_{11}$ and $G_{22}$ have minimum values. Moreover, at the observation angle $\phi = 180^\circ$, $G_{12}$ vanishes, and $G_{11}$ and $G_{22}$ have maximum values. Similar considerations can be made for the transmitted rays of higher order.

In Fig. 9 are compared the degree of linear polarization for single scattering of incident unpolarized light at normal incidence, $\alpha = 0^\circ$, by ray optics and wave theory. Although the approximation of ray optics seems to be adequate for small radii such as 10 $\mu$m for the scattered intensity (Fig. 4), a much larger average radius of at least 30 $\mu$m is necessary to account for the degree of linear polarization with good accuracy.

Figure 10 compares the degree of linear polarization at the oblique incident angles $\alpha = 45^\circ$ and $85^\circ$. In the case of $\alpha = 45^\circ$, the degree of linear polarization varies suddenly from $\phi = 178^\circ$ to $\phi = 180^\circ$ corresponding to the abrupt change of $G_{12}$ shown in Fig. 7. At $\alpha = 85^\circ$, there is close agreement among ray optics and wave theory. But at $\phi \approx 180^\circ$, wave theory becomes negatively intense with the increase of $\bar{a}$ compared with ray optics A. This is caused by the rainbow due to the $p = 10$ component, which is clearly seen in ray optics B.
The differences at the forward direction among ray optics and wave theory are attributed to the apparent smallness of the cylinders in view of ray optics. From Figs. 9 and 10, it can be thought that the approximation of ray optics for the degree of linear polarization becomes better as the oblique incident angle $\alpha$ increases. At the direction $\phi \approx 180^\circ$, the degree of linear polarization becomes strongly negative as $\alpha$ increases, but this negative polarization becomes weaker for $\alpha > 60^\circ$. This feature is explained as follows. The externally reflected rays at $\phi \approx 180^\circ$ are those for $\beta \approx 0^\circ$, $\tau_1 \approx \alpha$, and $\delta_1 \approx 0^\circ$, so that $e^{(0)}_1 \approx r_1$ and $e^{(0)}_2 \approx r_2$. The ratio of $|r_1|^2$ to $|r_2|^2$ is minimum, when the incident angle $\alpha$ coincides with the Brewster angle, i.e., $52.7^\circ$ in the case of $m = 1.31 - 0.0i$.

Figure 11 shows the $G_{44}$ element at normal incidence, $\alpha = 0^\circ$, for ray optics and wave theory. It is clear that the element $G_{44}$ coincides with the element $G_{33}$, because $M_3 = 0$ in Eq. (24) at $\alpha = 0^\circ$. The element $G_{44}$ can be approximated by $|A_1||A_2|$ at the forward direction, since the amplitudes $A_1$ and $A_2$, to which the diffracted light and the transmitted light of the first order contribute, are nearly in phase. Therefore, this element is nearly the same as the phase function $G = (|A_1|^2 + |A_2|^2)/2$, because $|A_1| \approx |A_2|$ at the forward direction. For $\phi \approx 80^\circ$, the amplitudes of externally reflected rays $A_1^{(0)}$ and $A_2^{(0)}$ become out of phase, because the incident
angle is beyond the Brewster angle, so that $G_{44}$ becomes negative at the direction where the externally reflected light dominates. In the vicinity of the rainbow angle $\phi = 135^\circ$, the amplitudes of the transmitted light of the second order $A_1^{(2)}$ and $A_2^{(2)}$ are in phase, and their values are large, so that $G_{44}$ becomes intensely positive.

Figure 12 shows the element $G_{43} (= -G_{34})$ at normal incidence for ray optics and wave theory. The relative disagreement among them is most conspicuous compared with other phase matrix elements. However, since this element approaches zero over the whole range of observation angles as the average radius increases, the rather remarkable disagreement is not so serious for practical purposes. This tendency becomes more prominent at $\alpha = 45$ and $85^\circ$. In this connection, we can add a further comment: in the limit of ray optics (ray optics B), the element $G_{43}$ vanishes regardless of $\alpha$ for a polydisperse system of isotropic dielectric cylinders, while for absorbing cylinders, the element $G_{43}$ by ray optics B does not vanish, and ray optics B is closer to wave theory than ray optics A ($m = 1.31-0.1i$, $\bar{a} = 10 \mu m$).

Figure 13 compares the efficiency factors for extinction at normal incidence, $\alpha = 0^\circ$, for ray optics and wave theory. There are slowly damping sinusoidal oscillations similar to those for spherical particles. In view of ray optics, an interference of the diffracted light and the transmitted light of the first order is responsible for this oscillatory feature. For $Q_{\text{ext}}$ at $\alpha = 45^\circ$, the period of the oscillation in $x$ is shorter, i.e., 10.3 for $\alpha = 0^\circ$ and 7.99 for $\alpha = 45^\circ$, but the patterns are similar to each other.

Figure 14 illustrates the extinction efficiency factors at $\alpha = 85^\circ$ for ray optics and wave theory. The variations of $Q_{\text{ext}}$ are not sinusoidal in this case but quasi-periodic. The difference between wave theory and ray optics A are larger compared with the case of $\alpha = 0^\circ$, but the positions of the ridges and troughs coincide fairly well with each other. These quasi-periodic variations may be explained in view of ray optics as follows. Because of the large oblique incident angle, the reflectivity is high for internal reflections, so that multiple internal reflection should be considered. Therefore not only the diffracted light and the transmitted light of the first order but also the transmitted light of the higher order interfere with each other, and this interference will generate the quasi-periodic oscillations.

If we include the phase factor $[-2k + (1 - q)/2]\pi/2$ in Eq. (19), there will be a discrepancy on the phase of the oscillation of $Q_{\text{ext}}$ between ray optics A and wave theory shown in Figs. 13 and 14. It is noted from Figs. 13 and 14 that the efficiency factor for extinction obtained from wave theory is larger than that obtained.
from ray optics A. This difference between wave theory and ray optics A seems to show effects of grazing reflection. But for smaller size parameters, the inapplicability of ray optics is also responsible for the difference in its own right. However, ray optics A will give the extinction efficiency factor more realistically than the anomalous diffraction approximation especially in the case of $\alpha = 85^\circ$. Because ray optics A takes into account as many factors as possible, except for the effects of grazing reflection, on the other hand, the anomalous diffraction approximation takes into account only the diffracted light and the transmitted light of the first order for refractive index $m$ close to unity. The behavior of $Q_{\text{ext}}$ by wave theory in Figs. 13 and 14, which are equivalent to Figs. 8 or 9 of Ref. 15, is similar to that of slender prolate spheroids (Ref. 19, Fig. 7).

In Table I, the extinction efficiency factors averaged over the size distribution [Eq. (44)] evaluated by ray optics A are compared with those by wave theory. The average extinction efficiency factors are asymptotic to 2.0 from top to bottom in each column.

To examine the effects of absorption, the imaginary part of the complex refractive index was increased by keeping the real part unchanged. Figure 15 shows the phase functions by ray optics and wave theory for the absorbing cylinder with $m = 1.31 - 0.1i$ ($\lambda = 0.7 \mu m$). The scattered light intensity consists mainly of the diffracted light and the externally reflected light at any oblique incident angles due to the strong absorption. Therefore, the relative differences among these three methods at the forward direction become more striking, but the absolute differences are nearly the same as those for the nonabsorbing case. On the other hand, there is better agreement among ray optics and wave theory at the observation angle $\phi \geq 90^\circ$ for the case of $\alpha = 0$ and $45^\circ$.

Figure 16 shows the efficiency factors for extinction, scattering, and absorption and the single scattering albedo $\omega$ for the absorbing cylinder with $m = 1.31 - 0.1i$ ($\lambda = 0.7 \mu m$). The efficiency factors for scattering $Q_{\text{sca}}$ oscillate for small $x$, say, $x \leq 30$. As $x$ becomes larger,
the transmitted light of the first order is almost exhausted, so that the oscillation due to interference disappears. The absorption efficiency factor $Q_{abs}$ computed from ray optics A or B increases monotonically with $x$ and approaches a limiting value, because almost all the light refracted into a large cylinder is absorbed. The radius at which $Q_{abs}$ becomes constant nearly coincides with that at which the oscillation of $Q_{sca}$ disappears. For large sizes, $Q_{ext}$ computed from wave theory and ray optics A approaches 2.0 with slightly decreasing and increasing tendencies, respectively. Although there are some differences in $Q_{ext}$ and $Q_{sca}$ among wave theory and ray optics, the ratio of $Q_{sca}$ to $Q_{ext}$ or the single scattering albedo $\omega$ is in good agreement.

Table II gives numerical values of the efficiency factors for extinction, scattering, and absorption, and the single scattering albedo averaged over the size distribution for the absorbing cylinder with $m = 1.31-0.0i$ ($\lambda = 0.7 \mu m$), computed by ray optics A and wave theory. The efficiency factors $Q_{ext}$ in this case have almost the same values as those of the nonabsorbing cylinders. Ray optics A becomes worse with an increase of $\alpha$ for any efficiency factors. On the other hand, it is fairly good for average single scattering albedo $\omega$, which is defined by $Q_{sca}/Q_{ext}$. The two polarization components of the efficiency factor for extinction $Q_{ext,l}$ and $Q_{ext,r}$ at $\alpha = 0^\circ$ computed from ray optics A are 1.9826 and 1.9843, and for scattering they are 1.0490 and 1.0077, respectively. On the other hand, those components of the efficiency factor for extinction computed from wave theory are 2.0457 and 2.0456, and for scattering they are 1.0871 and 1.0363, respectively. In the limit of the large size parameter, both polarization components of the extinction efficiency factor should approach 2.0, and the ratio of $Q_{sca}$ to $Q_{ext}$ or the single scattering albedo $\omega$ is in good agreement.

Therefore $E^{(0)}$ increases and $E^{(1)}$ decreases monotonically with $\alpha$ due to the increase of $r$. The energy of transmitted light of the $p$th order ($p \geq 2$) is generally expressed by the geometrical series

$$r^{p-1}(1-r)^2,$$

where the common ratio is $r$ ($\leq 1$). When $\alpha$ is relatively small where $r$ is also small, the increase of reflection term $r^{p-1}$ in Eq. (46) with a small increase of $r$ overcomes the decrease of transmission term $(1-r)^2$, so that the energies contained in the transmitted light of the higher order increase. On the other hand, when $\alpha$ becomes large where $r$ increases rapidly, the decrease of transmission term overcomes the increase of reflection term, so that $E^{(p)}$ ($p \geq 2$) decreases, and the convergence of the series becomes slow and a large number of terms is required. For a ray of light incident on any position of a cylinder, the situation is the same as the above. Variations of $E^{(p)}$, $E^{(p)}$, and $E^{(p)}$ for $p \geq 2$ with $\alpha$ are not so simple as $E^{(p)}$ but rather winding. $E^{(p)}$ and $E^{(p)}$ increases where $E^{(p)}$ decreases and vice versa. This means that by a rotation of the incident plane, the radiant energy is distributed among the different polarization components, but $E^{(p)}$, which is the average of these components, varies fairly simply as shown in Fig. 17.

<table>
<thead>
<tr>
<th>$\alpha = 0^\circ$</th>
<th>$\alpha = 45^\circ$</th>
<th>$\alpha = 85^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>RO(A)</td>
<td>WT</td>
<td>RO(A)</td>
</tr>
<tr>
<td>$Q_{ext}$</td>
<td>1.9834</td>
<td>2.0457</td>
</tr>
<tr>
<td>$Q_{sca}$</td>
<td>1.0283</td>
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<tr>
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<td>0.9840</td>
</tr>
<tr>
<td>$\omega$</td>
<td>0.5185</td>
<td>0.5190</td>
</tr>
</tbody>
</table>

The radius at which $Q_{abs}$ becomes constant nearly coincides with that at which the oscillation of $Q_{sca}$ disappears. For large sizes, $Q_{ext}$ computed from wave theory and ray optics A approaches 2.0 with slightly decreasing and increasing tendencies, respectively. Although there are some differences in $Q_{ext}$ and $Q_{sca}$ among wave theory and ray optics, the ratio of $Q_{sca}$ to $Q_{ext}$ or the single scattering albedo $\omega$ is in good agreement.

Table II gives numerical values of the efficiency factors for extinction, scattering, and absorption and single scattering albedo $\omega$ averaged over Size Distribution for $m = 1.31-0.0i$ ($\lambda = 0.7 \mu m$), computed by ray optics A and wave theory. The efficiency factors $Q_{ext}$ in this case have almost the same values as those of the nonabsorbing cylinders. Ray optics A becomes worse with an increase of $\alpha$ for any efficiency factors. On the other hand, it is fairly good for average single scattering albedo $\omega$, which is defined by $Q_{sca}/Q_{ext}$. The two polarization components of the efficiency factor for extinction $Q_{ext,l}$ and $Q_{ext,r}$ at $\alpha = 0^\circ$ computed from ray optics A are 1.9826 and 1.9843, and for scattering they are 1.0490 and 1.0077, respectively. On the other hand, those components of the efficiency factor for extinction computed from wave theory are 2.0457 and 2.0456, and for scattering they are 1.0871 and 1.0363, respectively. In the limit of the large size parameter, both polarization components of the extinction efficiency factor should approach 2.0, and the ratio of $Q_{sca}$ to $Q_{ext}$ or the single scattering albedo $\omega$ is in good agreement.

Therefore $E^{(0)}$ increases and $E^{(1)}$ decreases monotonically with $\alpha$ due to the increase of $r$. The energy of transmitted light of the $p$th order ($p \geq 2$) is generally expressed by the geometrical series

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VI. Conclusions

After the above comparison of ray optics and the exact electromagnetic wave theory for several quantities for single scattering by circular cylinders, it has become evident that the applicability of the approximation of ray optics is dependent on the orientation of cylinders relative to the incident light as well as their size parameters. The applicability of ray optics also depends on what quantity for single scattering (e.g., the phase function, the degree of linear polarization, the extinction efficiency factor, or the single scattering albedo) is considered. Although in this study we investigated applicability of ray optics for the size parameters up to \( \sim 270 \), much larger size parameters of the order of several hundreds seem to be needed for valid application of the ray optics approximation to all the single scattering quantities. The variations of phase matrix elements and scattering cross sections with the size parameter computed by ray optics A agreed with those by wave theory, but the method of ray optics A still requires a fairly long computation time. On the other hand, the approximation of ray optics B gives fairly accurate phase matrix elements for a polydisperse system, and a much less computer time is needed. For example, for \( m = 1.31-0.01 \) (\( \lambda = 0.7 \mu m \)) and \( d = 30 \mu m \) the computation time by ray optics B is only 0.4 \( \sim 4\% \) of the corresponding time by wave theory, whereas ray optics A requires about 20–160\% of wave theory. From an optical point of view, it may be useful to study further the solution of ray optics A, but from a point of practical use, e.g., in meteorology, ray optics B will provide a convenient means to investigate the phase matrix of other nonspherical particles like hexagonal cylinders.

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Appendix A

Applying Huygens's principle and Babinet's theorem (Ref. 5, p. 105), the disturbance \( u_s \) caused by diffraction about a circular cylinder is expressed as

\[
 u_s = -\frac{i}{\lambda} \sum_{-a}^{a} \int_{-\infty}^{\infty} \exp \left[ i(\omega t - kX \cos \alpha) + Z \sin \alpha \right] dYdZ. \tag{A1}
\]

Here \( u_s \) is the disturbance in the original wave, the term \( \omega t - kX \cos \alpha + Z \sin \alpha \) denotes the phase of the plane wave incident on the infinitely long slit at an oblique incident angle \( \alpha \), and \( \vec{r} \) is the distance from a point \( (O, Y, Z) \) in the slit to an observation point with sufficiently large \( R \), i.e.,

\[
 \vec{r} = R \left[ 1 - \frac{2Y}{R} \cos \phi \sin \phi + \frac{2Z}{R} \sin \phi + \frac{Y^2 + Z^2}{R^2} \right]^{1/2} \approx R - Y \cos \phi \sin \phi - Z \sin \phi + \frac{Z^2}{2R}. \tag{A2}
\]

Thus, the explicit expression for the amplitude can be obtained by direct integration of Eq. (A1) as

\[
 E_s = (2x \cos \alpha)^{1/2} \sin(x \cos \alpha \sin \phi) \frac{\frac{x}{x \cos \sin \phi}}{\pi k \vec{r}} \exp \left[ i(\omega t - kR - \frac{3x}{4}) \right] E_i. \tag{A3}
\]

Here the disturbances \( u_i \) and \( u_s \) are interpreted, respectively, as \( (\cos \phi)^{1/2}E_i \) and \( E_s \). Comparing Eq. (A3) with the definition of the amplitude matrix, we easily find that \( A_D \) is given by Eq. (4).

Appendix B

When a rainbow appears, the fraction of energy contained in the transmitted light of the order responsible for the rainbow is obtained by integrating the intensity over the observation angle; i.e.,

\[
 \omega^{\rho j} = \int_{0}^{\pi} |A^{\rho j}|^2 d\phi, \quad j = 1, 2, 3, 4. \tag{B1}
\]

The energy thus determined does not always agree with the fraction of energy contained originally in the transmitted light of that order \( E^{\rho j} \) because \( \omega^{\rho j} \) involves instability resulting from the fact that \( A^{\rho j} \) has an infinite value at an exact rainbow angle. The classical theory of geometrical optics fails also in the vicinity of the rainbow angle. Therefore, we intend to remove the instability of \( \omega^{\rho j} \) or \( Q_{sca} \) and to correct the intensity at an approximate rainbow angle \( \phi_R \) by replacing \( A^{\rho j} \) at \( \phi_R \) with

\[
 \eta^{\rho j} \exp[i\arg(A^{\rho j})], \quad \text{for } |\eta^{\rho j}|^2 \geq 0,
\]

\[
 0, \quad \text{for } |\eta^{\rho j}|^2 < 0, \tag{B2}
\]

where

\[
 \eta^{\rho j} = |A^{\rho j}|^2 - |E^{\rho j}|/\Delta \omega^{\rho j} \tag{B3}
\]

and \( \Delta \omega \) is the quadrature weight at \( \phi_R \). The sign of the factor \( |\eta^{\rho j}|^2 \) was often found to be negative for small \( x \), but it became almost positive for large \( x \).

References