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GENERALIZATION OF THE SPHERICAL HARMONIC METHOD TO RADIATIVE TRANSFER IN MULTI-DIMENSIONAL SPACE

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Abstract—The basic radiative transfer equation in three-dimensional space is expressed in terms of three commonly used coordinate systems, namely, Cartesian, cylindrical and spherical coordinates. The concept of a transformation matrix is applied to the transformation processes between the Cartesian system and two other systems. The spherical harmonic method is then applied to decompose the radiative transfer equation into a set of coupled partial differential equations for all three systems in terms of partial differential operators. By truncating the number of partial differential equations into four along with further mathematical analyses, we obtain a modified Helmholtz equation. For each coordinate system, analytical solutions in terms of infinite series are obtained whenever the equation is solvable by the technique of separation of variables with proper boundary conditions. Numerical computations are carried out for one dimensional radiative transfer to illustrate the applicability of the technique developed in the present study.

1. INTRODUCTION

Various research disciplines, such as atmospheric sciences, astronomy, nuclear engineering, engineering design, etc., are frequently involved with problems of radiative transfer. Generally, the physical behaviors of radiative transfer are summarized in the basic transfer equation or the Boltzmann equation. The inherent complexity of the equation leads to a wide variety of solution methods. Crosbie and Linsenbardt[1], in their study on the two-dimensional radiative transfer involving isotropic scattering, have presented a rather comprehensive overview on a number of methods that were used to solve the transfer equation under various physical situations. Bayazitoglu[2] has also provided a general survey on the development of more refined solution methods for the multi-dimensional transfer equation. Among these solution methods, the spherical harmonics method is perhaps the most tedious and yet elegant approach. It was first introduced by Jeans[3] in conjunction with astronomical radiative transfer problems. Later, Marshak[4] applied it to the Milne problem for a sphere. Mark[5] also generalized the spherical harmonics method in terms of a tensor operation to medium with cylindrical and spherical symmetry. Dave and Canosa[6] and Dave[7] presented a direct solution to the one-dimensional transfer equation by means of the spherical harmonics method. More recently, Liou and Ou[8] and Ou and Liou[9] also developed analytical and numerical schemes based on the first-order spherical harmonics expansion to solve the transfer of solar and infrared radiation in three-dimensional cloud layers. On the basis of the aforementioned studies, it would appear that the spherical harmonics method as applied to the radiative transfer equation may be generalized for arbitrary coordinate systems and dimensions.

In this paper we first apply the spherical harmonics method to the basic three-dimensional radiative transfer equation in terms of the three most commonly used coordinate systems (Cartesian, cylindrical, and spherical). The finite expansion for both intensity and phase function are inserted into the basic equation and a set of coupled partial differential equations are obtained by means of the orthogonal property of the spherical harmonics. We then formulate the first-order approximation together with all possible boundary conditions assuming that the model medium considered is subject to internal emission only. The number of partial differential equations in this case is reduced to four, and a modified Helmholtz equation is subsequently obtained. Analytical solutions in the form of series expansion are then derived whenever the requirements for a Sturm–Liouville type problem are satisfied. Examples of the computation based on the first-order approximation for the three coordinate systems in one-dimensional space are then presented to demonstrate the practicality of the method.

2. BASIC RADIATIVE TRANSFER EQUATIONS IN THREE FUNDAMENTAL COORDINATE SYSTEMS

The fundamental time-independent equation describing the transfer of monochromatic radiation in the earth's atmosphere can be written in the form

$$-\frac{1}{\sigma}(\mathbf{\Omega}\cdot\nabla)I(s,\mathbf{\Omega})+I(s,\mathbf{\Omega})=\frac{\tilde{\omega}}{4\pi}\int_{4\pi} I(s,\mathbf{\Omega}')P(\mathbf{\Omega},\mathbf{\Omega}')\,\mathrm{d}\Omega'+J(s,\mathbf{\Omega}), \tag{2.1}$$

where I denotes the monochromatic intensity of the scattered radiation, σ the extinction coefficient, $\mathbf{\Omega}$ a unit vector specifying the direction of scattering through a position vector s, $\tilde{\omega}$ the single-scattering albedo, and P the normalized scattering phase function. The source function J in the solar and thermal i.r. radiation regions, subject to the local thermodynamic equilibrium assumption, can be expressed separately as follows:

$$J(s,\mathbf{\Omega})=\begin{cases}(1-\tilde{\omega})B_\nu[T(s)] & \text{IR}\\ \frac{\tilde{\omega}}{4\pi}P(\mathbf{\Omega},\mathbf{\Omega}_0)\pi F\,\bar{e}^{-\int_s \sigma(s)\,\mathrm{d}s} & \text{SOL}\end{cases} \tag{2.2}$$

where $B_\nu(T)$ is the Planck function of temperature T at wavenumber ν, πF denotes the direct solar irradiance at the top of the atmosphere, and IR and SOL represent i.r. and solar radiation, respectively.

In Cartesian coordinates (x, y, z), Eq. (2.1) may be explicitly written in terms of the angular system as

$$\frac{1}{\sigma}\left(\sin\theta\cos\phi\frac{\partial}{\partial x}+\sin\theta\sin\phi\frac{\partial}{\partial y}+\cos\theta\frac{\partial}{\partial z}\right)I(x,y,z;\theta,\phi)+I(x,y,z;\theta,\phi)$$
$$=\frac{\tilde{\omega}}{4\pi}\int_0^{2\pi}\int_{-1}^{1} I(x,y,z;\theta',\phi')P(\theta,\phi;\theta',\phi')\,\mathrm{d}\cos\theta'\,\mathrm{d}\phi'+J(s,y,z;\theta,\phi). \tag{2.3}$$

In writing this equation, we note that the directional cosines $\Omega_x=\sin\theta\cos\phi$, $\Omega_y=\sin\theta\sin\phi$, and $\Omega_z=\cos\theta$.

In order to express Eq. (2.1) in the cylindrical and spherical coordinates, it is necessary to perform the coordinate transformation for the directional cosines. We define the transformation matrix $\boldsymbol{T}$ as

$$\begin{bmatrix}\Omega_1'\\ \Omega_2'\\ \Omega_3'\end{bmatrix}=\boldsymbol{T}\begin{bmatrix}\Omega_1\\ \Omega_2\\ \Omega_3\end{bmatrix}. \tag{2.4}$$

Since the direction cosine between two vectors is the dot product of the unit vectors along these two vectors, it can be shown that the direction cosine vector in the prime system is given by

$$i=\boldsymbol{e}_i\cdot\begin{bmatrix}\Omega_1\\ \Omega_2\\ \Omega_3\end{bmatrix}=[e_{i1}e_{i2}e_{i3}]\begin{bmatrix}\Omega_1\\ \Omega_2\\ \Omega_3\end{bmatrix}, \tag{2.5}$$

where $\boldsymbol{e}_i$ is the unit vector in the direction of the i(th) axis of the prime system and can be expressed in terms of the bare vectors of the original system. It follows that the transformation matrix is

$$\boldsymbol{T}=\begin{bmatrix}e_{11} & e_{12} & e_{13}\\ e_{21} & e_{22} & e_{23}\\ e_{31} & e_{32} & e_{33}\end{bmatrix}. \tag{2.6}$$

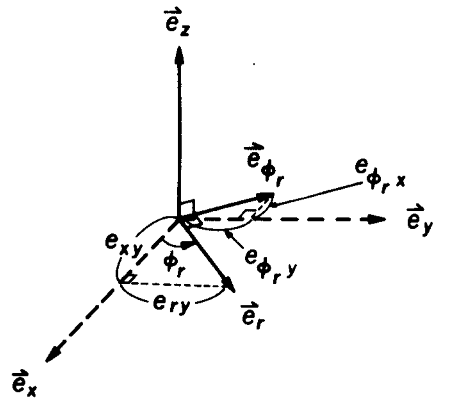

Fig. 1. The transformation of the base vectors in Cartesian coordinates to those in cylindrical coordinates.

In reference to Fig. 1, we now perform the transformation of the base vectors in Cartesian coordinates to those in cylindrical coordinates (r, ϕ_r, z) by applying Eq. (2.6). Thus,

$$\left.\begin{aligned} e_r &= e_{rx}e_x + e_{ry}e_y + e_{rz}e_z, \\ e_{\phi_r} &= e_{\phi_r x}e_x + e_{\phi_r y}e_y + e_{\phi_r z}e_z, \\ e_z &= e_{zx}e_x + e_{zy}e_y + e_{zz}e_z. \end{aligned}\right\} \tag{2.7}$$

In this case, e_{ij} can easily be determined from the polar angle ϕ_r and the transformation matrix is simply given by

$$T_{\text{car-cyl}} = \begin{bmatrix} \cos\phi_r & \sin\phi_r & 0 \\ -\sin\phi_r & \cos\phi_r & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.8}$$

In a similar manner, the transformation matrix involving Cartesian and spherical coordinates $(\rho, \theta_\rho, \phi_\rho)$ is

$$T_{\text{car-sph}} = \begin{bmatrix} \cos\phi_\rho \sin\theta_\rho & \sin\phi_\rho \sin\theta_\rho & \cos\theta_\rho \\ -\cos\theta_\rho \cos\phi_\rho & -\cos\theta_\rho \sin\phi_\rho & \sin\theta_\rho \\ -\sin\phi_\rho & \cos\phi_\rho & 0 \end{bmatrix}. \tag{2.9}$$

Using Eqs. (2.8) and (2.9), the basic radiative transfer equation in cylindrical and spherical coordinates may now be written, respectively, as follows:

$$\begin{aligned} &\frac{1}{\sigma}\left[\cos(\phi-\phi_r)\sin\theta\frac{\partial}{\partial r} + \sin(\phi-\phi_r)\sin\theta\frac{\partial}{r\partial\phi_r} + \cos\theta\frac{\partial}{\partial z}\right]I(r,\phi_r,z;\theta,\phi) + I(r,\phi_r,z;\theta,\phi) \\ &= \frac{\tilde{\omega}}{4\pi}\int_1^2\int_{-1}^1 I(r,\phi_r,z;\theta',\phi')P(\theta,\phi;\theta',\phi')\,\mathrm{d}\cos\theta'\,\mathrm{d}\phi' + J(r,\phi_r,z;\theta,\phi), \end{aligned} \tag{2.10}$$

$$\begin{aligned} &\frac{1}{\sigma}\Big\{[\sin\theta\sin\theta_\rho\cos(\phi-\phi_\rho)+\cos\theta\cos\theta_\rho]\frac{\partial}{\partial\rho}[\sin\theta\cos\theta_\rho\cos(\phi-\phi_\rho)-\cos\theta\sin\theta_\rho]\frac{\partial}{\rho\partial\theta_\rho} \\ &\quad + [\sin\theta\sin(\phi-\phi_\rho)]\frac{\partial}{\rho\sin\theta_\rho\partial\phi_\rho}\Big\}I(\rho,\theta_\rho,\phi_\rho;\theta,\phi) + I(\rho,\theta_\rho,\phi_\rho;\theta,\phi) \\ &= \frac{\tilde{\omega}}{4\pi}\int_0^{2\pi}\int_{-1}^1 I(\rho,\theta_\rho,\phi_\rho;\theta,\phi)P(\theta,\phi;\theta',\phi')\,\mathrm{d}\cos\theta'\,\mathrm{d}\phi' + J(\rho,\theta_\rho,\phi_\rho;\theta,\phi). \end{aligned} \tag{2.11}$$

3. DECOMPOSITION OF THE FUNDAMENTAL RADIATIVE TRANSFER EQUATIONS IN THREE COORDINATE SYSTEMS

In the previous section, we expressed the fundamental integro-differential transfer equation in three coordinate systems. We would like to demonstrate in this section that Eqs. (2.3), (2.10), and (2.11) can be decomposed into an identical set of partial differential equations for all systems.

Let $\mathbf{\Omega}^t_{\text{car}}$ and $\mathbf{T}^t$ be the transposes of the row vector in Cartesian coordinates $\mathbf{\Omega}_{\text{car}}$ and the matrix $\mathbf{T}$, respectively. We may then define the row vector $\mathbf{\Omega} = \mathbf{\Omega}^t_{\text{car}}\mathbf{T}^t$ which is valid for any coordinate system. Utilizing this definition Eq. (2.1) may be written as

$$-\frac{1}{\sigma}[\mathbf{\Omega}^t_{\text{car}} \cdot (\mathbf{T}^t \cdot \nabla)]I(s, \mathbf{\Omega}) + I(s, \mathbf{\Omega}) = \frac{\tilde{\omega}}{4\pi}\int_{4\pi} I(s, \mathbf{\Omega}')P(\mathbf{\Omega}, \mathbf{\Omega}')\, d\Omega' + J(s). \tag{3.1}$$

At this point, we introduce the spherical-harmonics expansion for the scattered phase function and intensity in a manner defined by Case and Zweifel[10] and Liou and Ou[8] as follows:

$$P(\mathbf{\Omega}, \mathbf{\Omega}') = \sum_{l=0}^{N}\sum_{m=-l}^{l} \tilde{\omega}_l Y_l^m(\mathbf{\Omega}) Y_l^{m*}(\mathbf{\Omega}'), \tag{3.2}$$

$$I(s, \mathbf{\Omega}) = \sum_{l=0}^{N}\sum_{m=-l}^{l} I_l^m(s) Y_l^m(\mathbf{\Omega}), \tag{3.3}$$

where N denotes the number of terms in the spherical-harmonics expansion and the spherical harmonics is defined by

$$Y_l^m(\theta, \phi) = (-1)^{(m+|m|)/2}\left[\frac{(l-|m|!}{(l+|m|)!}\right]^{1/2} P_l^{|m|}(\cos\theta)\, e^{im\phi}, \tag{3.4}$$

here P_l^m is the associated Legendre polynomials and the complex conjugate value of the spherical harmonics is given by

$$Y_l^{m*}(\theta, \phi) = \frac{1}{(-1)^m} Y_l^{-m}(\mathbf{\Omega}), \tag{3.5}$$

such that

$$\int_{4\pi} Y_l^m(\mathbf{\Omega}) Y_\alpha^{\beta*}(\mathbf{\Omega})\, d\Omega = \frac{4\pi}{2l+1}\delta_l^\alpha \delta_m^\beta, \tag{3.6}$$

with δ_l^α and δ_m^β being the Kronecker delta functions. Inserting Eqs. (3.2) and (3.3) into Eq. (3.1) we obtain

$$-\frac{1}{\sigma}[\mathbf{\Omega}^t_{\text{car}} \cdot (\mathbf{T}^t \cdot \nabla)]\sum_{l=0}^{N}\sum_{m=-l}^{l} I_l^m(s) Y_l^m(\mathbf{\Omega}) = -\sum_{l=0}^{N}\sum_{m=-l}^{l} \gamma_l I_l^m(s) Y_l^m(\mathbf{\Omega}) + J(s), \tag{3.7}$$

where

$$\gamma_l = 1 - \tilde{\omega}\tilde{\omega}_l/(2l+1). \tag{3.8}$$

Furthermore, upon carrying out the following operation

$$\int_{4\pi} \text{Eq. (3.7)} \times Y_\alpha^{\beta*}(\mathbf{\Omega})\, d\Omega, \quad \alpha = 0, 1, \ldots, N,$$

$$\beta = -\alpha, \ldots, \alpha,$$

we find

$$-\frac{1}{\sigma}\sum_{l=0}^{N}\sum_{m=-l}^{l}\int_{4\pi} d\Omega[\boldsymbol{\Omega}_{car}\cdot(T^t\cdot\nabla)]Y_l^m(\boldsymbol{\Omega})Y_\alpha^{\beta *}(\boldsymbol{\Omega})I_l^m(s)$$

$$= -\gamma_\alpha I_\alpha^{\beta}(s)\frac{4\pi}{2\alpha+1}+\int_{4\pi} J(s,\boldsymbol{\Omega})Y_\alpha^{\beta *}(\boldsymbol{\Omega})\,d\Omega. \tag{3.9}$$

The general recursion relationships involving $Y_l^m(\boldsymbol{\Omega})\boldsymbol{\Omega}_{car}$ may be expressed in the form

$$\begin{aligned} Y_l^m(\boldsymbol{\Omega})\boldsymbol{\Omega}_{car} = &\left\{\frac{2l+3}{2l+1}[-A(l,m)Y_{l+1}^{m+1}(\boldsymbol{\Omega})+B(l,m)Y_{l+1}^{m-1}(\boldsymbol{\Omega})]\right.\\ &\left.+\frac{2l-1}{2l+1}[C(l,m)Y_{l-1}^{m+1}(\boldsymbol{\Omega})-D(l,m)Y_{l-1}^{m-1}(\boldsymbol{\Omega})]\right\}e_x\\ &+i\left\{\frac{2l+3}{2l+1}[A(l,m)Y_{l+1}^{m+1}(\boldsymbol{\Omega})+B(\Omega,m)Y_{\Omega+1}^{m-1}(\boldsymbol{\Omega})]\right.\\ &\left.-\frac{2l-1}{2l+1}[C(l,m)Y_{l-1}^{m+1}(\boldsymbol{\Omega})+D(\Omega,m)Y_{\Omega-1}^{m-1}(\boldsymbol{\Omega})\right\}e_y\\ &+\frac{2l+3}{2l-1}E(l,m)Y_{l+1}^{m}(\boldsymbol{\Omega})+\frac{2l-1}{2l+1}F(l,m)Y_{l-1}^{m}(\boldsymbol{\Omega})\Big\}e_z, \end{aligned} \tag{3.10}$$

where

$$\left.\begin{aligned} A(l,m) &= [(l+m+1)(l+m+2)]^{1/2}/[2(2l+3)],\\ B(l,m) &= [(l-m+1)(l-m+2)]^{1/2}/[2(2l+3)],\\ C(l,m) &= [(l-m-1)(l-m)]^{1/2}/[2(2l-1)],\\ D(l,m) &= [(l+m-1)(l+m)]^{1/2}/[2(2l-1)],\\ E(l,m) &= [(l-m+1)(l+m+1)]^{1/2}/(2l+3),\\ F(l,m) &= [(l-m)(l+m)]^{1/2}/(2l-1). \end{aligned}\right\} \tag{3.11}$$

We note that these coefficients may be related through the variations of the indices l and m. We now substitute Eq. (3.10) into Eq. (3.9) and make use of the orthogonal property denoted in Eq. (3.6). We find

$$\frac{1}{\sigma}\left\{(T^t\cdot\nabla)^t\cdot\begin{bmatrix} X+Y\\ i(X-Y)\\ Z\end{bmatrix}\right\}+\gamma_\alpha I_\alpha^{\beta}(s)=\frac{2\alpha+1}{4\pi}\int_{4\pi} J(s,\boldsymbol{\Omega})Y_\alpha^{\beta *}(\boldsymbol{\Omega})\,d\Omega, \tag{3.12}$$

where

$$\begin{aligned} X &= -A(\alpha,\beta)I_{\alpha+1}^{\beta+1}(s)+C(\alpha,\beta)I^{\beta+1}(s),\\ Y &= B(\alpha,\beta)I_{\alpha+1}^{\beta-1}(s)-D(\alpha,\beta)I_{\alpha-1}^{\beta-1}(s),\\ Z &= E(\alpha,\beta)I_{\alpha+1}^{\beta}(s)+F(\alpha,\beta)I_{\alpha-1}^{\beta}(s), \end{aligned} \tag{3.13}$$

and the operator matrix may be defined by

$$(T^t\nabla)^t=\left[\frac{L_+ + L_-}{2}\;\frac{L_+ - L_-}{2i}\;L\right], \tag{3.14}$$

so that Eq. (3.12) becomes

$$\frac{1}{\sigma}(L_+X + L_-Y + LZ) + \gamma_\alpha I_\alpha^\beta(s) = \frac{2\alpha+1}{4\pi}\int_{4\pi} J(s,\mathbf{\Omega})Y_\alpha^{\beta*}(\mathbf{\Omega})\,d\Omega. \tag{3.15}$$

In Eqs. (3.14) and (3.15), the partial differential operators in Cartesian coordinates are given by

$$\left.\begin{aligned} L_- &= \frac{\partial}{\partial x} - i\frac{\partial}{\partial y},\\ L &= \frac{\partial}{\partial z},\\ L_+ &= \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}. \end{aligned}\right\} \tag{3.16}$$

For cylindrical coordinates, we find

$$\left.\begin{aligned} L_- &= e^{-i\phi_r}\left(\frac{\partial}{\partial r} - i\frac{\partial}{r\partial\phi_r}\right),\\ L &= \frac{\partial}{\partial z},\\ L_+ &= e^{i\phi_r}\left(\frac{\partial}{\partial r} + i\frac{\partial}{r\partial\phi_r}\right). \end{aligned}\right\} \tag{3.17}$$

For spherical coordinates, they are

$$\left.\begin{aligned} L_- &= e^{-i\phi_\rho}\left[\left(\sin\theta_\rho\frac{\partial}{\partial\rho} + \cos\theta_\rho\frac{\partial}{\rho\partial\theta_\rho}\right) - i\frac{\partial}{\rho\sin\theta_\rho\partial\phi_\rho}\right],\\ L &= \cos\theta_\rho\frac{\partial}{\partial\rho} - \sin\theta_\rho\frac{\partial}{\rho\partial\theta_\rho},\\ L_+ &= e^{i\phi_\rho}\left[\left(\sin\theta_\rho\frac{\partial}{\partial\rho} + \cos\theta_\rho\frac{\partial}{\rho\partial\theta\rho}\right) + i\frac{\partial}{\rho\sin\theta_\rho\partial\phi_\rho}\right]. \end{aligned}\right\} \tag{3.18}$$

Equation (3.15) represents the general decomposed radiative transfer equation with the linear differential operators given in Eqs. (3.16)–(3.18).

4. FIRST ORDER SPHERICAL-HARMONIC APPROXIMATIONS AND SOLUTIONS

Equation (3.12) represents a set of partial differential equations which, in principle, may be solved by means of numerical methods. However, the simplest approach to the three-dimensional radiative transfer problem would be to truncate the spherical-harmonics expansion for the scattering phase function and intensity at the second term, i.e., letting $N = 1$, where N is the maximum lower index in the truncated series. Using this first-order approximation, which is also referred to as the diffusion approximation, we obtain the following four partial differential equations:

$$\frac{1}{\sigma}\left[\frac{1}{3}LI_1^0(s) + \frac{\sqrt{2}}{6}L_-I_1^{-1}(s) - \frac{\sqrt{2}}{6}L_+I_1^1(s)\right] + \gamma_0 I_0^0(s) = \frac{1}{4\pi}\int_{4\pi} J(s,\mathbf{\Omega})\,d\Omega, \tag{4.1}$$

$$\frac{\sqrt{2}}{2\sigma}L_+I_0^0(s) + \gamma_1 I_1^{-1}(s) = \frac{3}{4\pi}\int_{4\pi} J(s,\mathbf{\Omega})Y_1^{-1*}(\mathbf{\Omega})\,d\Omega, \tag{4.2}$$

$$\frac{1}{\sigma} L I_0^0(s) + \gamma_1 I_1^0(s) = \frac{3}{4\pi} \int_{4\pi} J(s, \mathbf{\Omega}) Y_1^{0*}(\mathbf{\Omega})\, \mathrm{d}\Omega, \tag{4.3}$$

$$-\frac{\sqrt{2}}{2\sigma} L_- I_0^0(s) + \gamma_1 I_1^1(s) = \frac{3}{4\pi} \int_{4\pi} J(s, \mathbf{\Omega}) Y_1^{1*}(\mathbf{\Omega})\, \mathrm{d}\Omega. \tag{4.4}$$

Substitution of Eqs. (4.2)–(4.4) into Eq. (4.1) yields

$$\begin{aligned} &\left(L^2 + \frac{1}{2} L_+ L_- + \frac{1}{2} L_- L_+\right) I_0^0(s) - 3\gamma_0\gamma_1\sigma^2 I_0^0(s) \\ &\quad - \frac{3\sigma}{4\pi} \int_{4\pi} \left[Y_1^{0*}(\mathbf{\Omega}) L + \frac{\sqrt{2}}{2} Y_1^{-1*}(\mathbf{\Omega}) L_- - \frac{\sqrt{2}}{2} Y_1^{1*}(\mathbf{\Omega}) L_+ \right] J(s, \mathbf{\Omega})\, \mathrm{d}\Omega \\ &= -\frac{3\gamma_1\sigma^2}{4\pi} \int_{4\pi} J(s, \mathbf{\Omega})\, \mathrm{d}\Omega, \end{aligned} \tag{4.5}$$

where γ_0 and γ_1 are defined in Eq. (3.8) and the extinction coefficient σ is assumed to be constant.

By virtue of the definitions of the partial differential operators given in Eqs. (3.16)–(3.18), it can be shown that

$$LL + \frac{1}{2} L_+ L_- + \frac{1}{2} L_- L_+ = \nabla^2, \tag{4.6}$$

where the Laplacian operator ∇^2 in the three basic coordinate systems is given by

$$\nabla^2_{\text{car}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \tag{4.7a}$$

$$\nabla^2_{\text{cyl}} = \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{\partial^2}{r^2 \partial \phi_r^2} + \frac{\partial^2}{\partial z^2}, \tag{4.7b}$$

$$\nabla^2_{\text{sph}} = \frac{1}{\rho^2} \frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\theta_\rho} \frac{\partial}{\partial \theta_\rho}\left(\sin\theta_\rho \frac{\partial}{\partial \theta_\rho}\right) + \frac{1}{\rho^2 \sin^2\theta_\rho} \frac{\partial^2}{\partial \phi_\rho^2}. \tag{4.7c}$$

In addition, we also find

$$Y_1^{0*}(\mathbf{\Omega}) L + \frac{\sqrt{2}}{2} Y_1^{-1*}(\mathbf{\Omega}) L_- - \frac{\sqrt{2}}{2} Y_1^{1*}(\mathbf{\Omega}) L_+ = \mathbf{\Omega} \cdot \nabla = \frac{\mathrm{d}}{\mathrm{d}s}. \tag{4.8}$$

Utilizing this simplification and applying the source function given in Eq. (2.2), analytic integrations can be performed for the integral term in Eq. (4.5). Thus, Eq. (4.5) can now be written in the form

$$\nabla^2 I_0^0(s) - 3\gamma_0\gamma_1\sigma^2 I_0^0(s) = \begin{cases} -\dfrac{3\sigma^2 \tilde{\omega} F}{4} (\gamma_1 + \tilde{\omega}_1)\, \mathrm{e}^{(-\int \sigma\, \mathrm{d}s)} \mathbf{\Omega}_0 & \text{SOL} \\ -3\sigma^2 \gamma_1 \gamma_0 B_\nu(T) & \text{IR} \end{cases} \tag{4.9}$$

Equation (4.9) represents an inhomogeneous modified Helmholtz equation whose solution depends on the boundary conditions imposed.

To seek a solution for Eq. (4.9), we assume that there is no inward diffuse intensity on the boundary of the medium. This is the so-called vacuum boundary condition, which may be expressed as

$$I_{\text{in}}(\mathbf{\Omega}) = 0 \text{ at boundary}, \tag{4.10}$$

where the subscript in denotes inward.

As pointed out by Marshak[4] and more recently by Dave and Canosa[6], Eq. (4.10) can not be satisfied exactly when mathematical solutions are derived by virtue of a finite expansion of the intensity. Thus, an approximate form of Eq. (4.10) is required in order to properly solve the partial differential equations derived from the basic radiative transfer equation. From Eq. (4.10) and in conjunction with the spherical harmonics approximations used in this study, we may write

$$\int_{\Omega} Y_l^m(\mathbf{\Omega}) I_{\text{in}}(\mathbf{\Omega})\, d\Omega = 0, \qquad l = \text{odd}, \tag{4.11}$$

where the domain of integration is the inner half plane. Substituting Eq. (3.3) into Eq. (4.11), we find

$$\sum_{l'=0}^{N} \sum_{m=-l'}^{l'} \left[I_{l'}^{m'}(s) \int_{\Omega} Y_l^m(\mathbf{\Omega}) Y_{l'}^{m'}(\mathbf{\Omega})\, d\Omega \right] = 0. \tag{4.12}$$

It should be noted that the domain of integration is a half hemisphere so that the orthogonal properties of spherical-harmonics are no longer applicable. Consequently, the analytical integration must be carried out for each pair of (l, m) and (l', m') at each location on the boundary. Since the solution of Eq. (4.9) depends on the specific geometry involved and its associated boundary conditions, it is not possible to seek a generalized solution form for all the three coordinate systems. Hence, we must derive separate solutions for Eq. (4.9) subject to Eq. (4.11) for each coordinate system.

(a) *Cartesian coordinates*

In Cartesian coordinates, Eq. (4.9) may be written in the form

$$\frac{\partial^2 I_0^0}{\partial x^2} + \frac{\partial^2 I_0^0}{\partial y^2} + \frac{\partial^2 I_0^0}{\partial z^2} - \Lambda^2 I_0^0 = \begin{cases} -\dfrac{3\sigma^2 \tilde{\omega} F}{4}(\gamma_1 + \tilde{\omega}_1)\, e^{-(z-z_0)/\cos\theta_0} & \text{SOL}, \\ -\Lambda^2 B_\nu(T) & \text{IR}, \end{cases} \tag{4.13}$$

where $\Lambda^2 = -3\sigma^2\gamma_1\gamma_0$, and z_0 is a reference value on the boundary.

To obtain the boundary conditions in this case, we zone the rectangle into six surfaces, as shown in Fig. 2, such that each surface is represented by a single algebraic equation, i.e., $x = 0$,

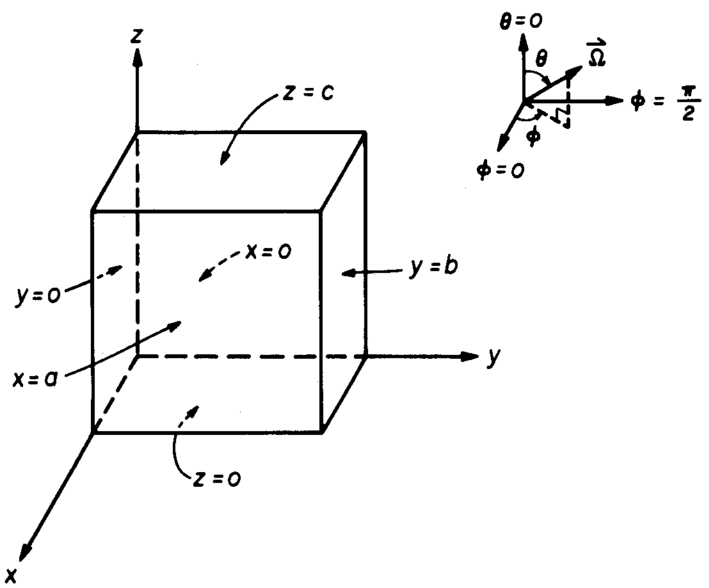

Fig. 2. A rectangular cubic element, where each surface plane is represented by a single algebraic equation and the angular notation is also illustrated.

$x = a$, $y = 0$, $y = b$, $z = 0$, and $z = c$, where a, b, and c are the lengths of the rectangle. Table 1 lists the domain of integration for the six surfaces with θ and ϕ being the zenith and azimuthal angles, respectively. After carrying out the integrations, we obtain the following six boundary conditions:

$$\left.\begin{aligned}
&I_0^0 + \frac{2}{3} I_1^0 = 0 && \text{at } z = 0,\\
&I_1^0 - \frac{2}{3} I_1^0 = 0 && \text{at } z = c,\\
&I_0^0 - \frac{\sqrt{2}}{3} i(I_1^{-1} + I_1^1) = 0 && \text{at } y = 0,\\
&I_0^0 + \frac{\sqrt{2}}{3} i(I_1^1 + I_1^{-1}) = 0 && \text{at } y = b,\\
&I_1^0 - \frac{\sqrt{2}}{3} (I_1^1 - I_1^{-1}) = 0 && \text{at } x = 0,\\
&I_0^0 + \frac{\sqrt{2}}{3} (I_1^1 - I_1^{-1}) = 0 && \text{at } x = a.
\end{aligned}\right\} \tag{4.14}$$

In order to express Eq. (4.14) in terms of I_1^0, it is necessary to utilize Eq. (3.16) and Eqs. (4.2)–(4.4) which involve the source function J. The source function in the solar case depends on the incident solar angle (θ_0, ϕ_0) and the coordinate values at the boundary (x_0, y_0, z_0). Thus, in general, Eq. (4.13) cannot be solved analytically and the solution is only available through numerical means.[9,11] However, when the sun is in an overhead position ($\cos\theta_0 = 1$), the analytical solution may be obtained (Ou and Liou, 1980). In the thermal infrared case, using the relations denoted in Eqs. (4.2)–(4.4), we obtain the boundary conditions in terms of I_0^0 as follows:

$$\left.\begin{aligned}
&\frac{\partial I_0^0}{\partial z} - hI_0^0 = 0 && \text{at } z = 0,\\
&\frac{\partial I_0^0}{\partial z} + hI_0^0 = 0 && \text{at } z = a,\\
&\frac{\partial I_0^0}{\partial y} - hI_1^0 = 0 && \text{at } y = 0,\\
&\frac{\partial I_0^0}{\partial y} + hI_0^0 = 0 && \text{at } y = b,\\
&\frac{\partial I_0^0}{\partial x} - hI_0^0 = 0 && \text{at } x = 0,\\
&\frac{\partial I_0^0}{\partial x} + hI_0^0 = 0 && \text{at } x = b,
\end{aligned}\right\} \tag{4.15}$$

where $h = 3\sigma\gamma_1/2$. The standard procedure for the method of separation of variables [8,12] may then be used to solve Eqs. (4.13) and Eq. (4.15). The general form of solution for I_0^0 is a double infinite series in the form

$$I_0^0(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (C_+ e^{\zeta_{nm} z} + C_- e^{-\zeta_{nm} z} + t_{nm}/\zeta_{nm}^2) U_n(x) V_m(y), \tag{4.16}$$

Table 1. The domain of integration in Cartesian coordinates.

Surface	θ	φ
x=0	0-π	$-\frac{\pi}{2}$ - $\frac{\pi}{2}$
x=b	0-π	$\frac{\pi}{2}$ - $\frac{3\pi}{2}$
y=0	0-π	0-π
y=b	0-π	π-2π
z=0	0-$\frac{\pi}{2}$	0-2π
z=a	$\frac{\pi}{2}$ - π	0-2π

where

$$\zeta_{nm}^2 = \lambda_n^2 + \lambda_m^2 + \Lambda^2,$$

$$t_{nm} = \frac{\Lambda^2 B_\nu(T)}{f_n g_m} \int_0^b \int_0^b U_n(x) V_m(y)\, dx\, dy,$$

$$f_n = \int_0^b U_n^2(x)\, dx,$$

$$g_m = \int_0^b V_m^2(y)\, dy,$$

$$U_n(x) = \cos(\lambda_n x) + \frac{h}{\lambda_n} \sin(\lambda_n x),$$

$$V_m(y) = \cos(\lambda_m y) + \frac{h}{\lambda_m} \sin(\lambda_m y).$$

The eigenvalues λ_n and λ_m can be found by solving the equation

$$2 \cot \lambda_l b = \frac{\lambda_l}{h} - \frac{h}{\lambda_l}, \quad \begin{cases} l = n \text{ or } m \\ l = 1, 2, \ldots \end{cases}$$

and C_+ and C_- are constants to be determined from the boundary conditions given by Eq. (4.15).

(b) *Cylindrical coordinates*

In cylindrical coordinates, Eq. (4.9) takes the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial I_0^0}{\partial r}\right) + \frac{\partial^2 I_0^0}{r^2 \partial \phi_r^2} + \frac{\partial^2 I_0^0}{\partial z^2} - \Lambda^2 I_0^0$$

$$= \begin{cases} -\dfrac{3\sigma^2 \tilde{\omega} F}{4}(\nu_1 + \tilde{\omega}_1) \exp\{-\sigma[(r_2^2 - r^2 \sin^2 \phi_r)^{1/2} - r \cos \phi_r] \sin \phi_0\} & \text{SOL}, \\ -\Lambda^2 B_\nu(T) & \text{IR}, \end{cases} \quad (4.17)$$

where the exponential term of the source function in the solar case is derived based on the assumption that the incoming solar beam is in the direction of $\phi_r = 0$, and r_2 is the radius of the cylinder. An analytical solution for the solar case is again impossible, because the inhomogeneous boundary conditions do not conform with the specifications of a Sturm–Liouville type problem.[13] However, in the IR case, it is possible to derive analytic solutions under certain circumstances, and we shall concentrate our effort in this case.

As shown in Fig. 3, for a cylindrical element bounded by six surfaces ($r = r_1$, $r = r_2$, $\phi_r = \phi_1$, $\phi_r = \phi_2$, $z = z_1$, and $z = z_2$) subject to monochromatic interior IR emission without any incoming radiation, Eq. (4.17) can be used as the governing equation. With a similar approach to obtain boundary conditions as in the case of Cartesian coordinates, we derive the following boundary conditions based on Eq. (4.12) as follows:

$$\left.\begin{aligned}
&I_0^0 + \frac{\sqrt{2}}{3}[(I_1^{-1} - I_1^1)\cos\phi_r - i(I_1^{-1} + I_1^1)\sin\phi_r] = 0 && \text{at } r = r_1,\\
&I_0^0 - \frac{\sqrt{2}}{3}[(I_1^{-1} - I_1^1)\cos\phi_r - i(I_1^{-1} + I_1^1)\sin\phi_r] = 0 && \text{at } r = r_2,\\
&I_0^0 + \frac{\sqrt{2}}{3}[-(I_1^{-1} - I_1^1)\sin\phi_r - i(I_1^{-1} + I_1^1)\cos\phi_r] = 0 && \text{at } \phi_r = \phi_1,\\
&I_0^0 - \frac{\sqrt{2}}{3}[-(I_1^{-1} - I_1^1)\sin\phi_r - i(I_1^{-1} + I_1^1)\cos\phi_r] = 0 && \text{at } \phi_r = \phi_2,\\
&I_0^0 + \frac{2}{3}I_1^0 = 0 && \text{at } z = z_1,\\
&I_0^0 - \frac{2}{3}I_1^0 = 0 && \text{at } z = z_2.
\end{aligned}\right\} \tag{4.18}$$

The domains of integration for obtaining Eq. (4.18) are listed in Table 2. Based on Eqs. (4.2)–(4.4) with the application of Eq. (3.17) for the definitions of differential operators, Eq. (4.18) can be further reduced to the forms in I_0^0 only, i.e.,

$$\left.\begin{aligned}
&\frac{\partial I_1^0}{\partial r} \pm hI_0^0 = 0 && \text{at } \begin{cases} r = r_1(-),\\ r = r_2(+);\end{cases}\\
&\frac{\partial I_0^0}{r\partial\phi_r} \pm hI_0^0 = 0 && \text{at } \begin{cases} \phi_r = \phi_1(-),\\ \phi_r = \phi_2(+);\end{cases}\\
&\frac{\partial I_0^0}{\partial z} \pm hI_0^0 && \text{at } \begin{cases} z = z_1(-),\\ z = z_2(+).\end{cases}
\end{aligned}\right\} \tag{4.19}$$

We notice that the second set of equations has a term $\partial I/r\partial\phi_r$, which causes the boundary conditions to deviate from the requirement of the Sturm–Liouville type problem, i.e., the orthogonal property of the separated ϕ_r-dependent functions cannot be established. Thus, an analytical solution at this point does not exist, but certain numerical approaches are possible.

Table 2. The domain of integration in cylindrical coordinates.

Surface	θ	ϕ
$r=r_1$	$0-\pi$	$\frac{\pi}{2}+\phi_r - \frac{3\pi}{2}+\phi_r$
$r=r_2$	$0-\pi$	$\frac{\pi}{2}-\phi_r - \frac{\pi}{2}+\phi_r$
$\phi_r=\phi_1$	$0-\pi$	$\phi_r - \phi_r + \pi$
$\phi_r=\phi_2$	$0-\pi$	$\phi_r + \pi - \phi_r + 2\pi$
$z=z_1$	$0-\frac{\pi}{2}$	$0 - 2\pi$
$z-z_2$	$\frac{\pi}{2}-\pi$	$0 - 2\pi$

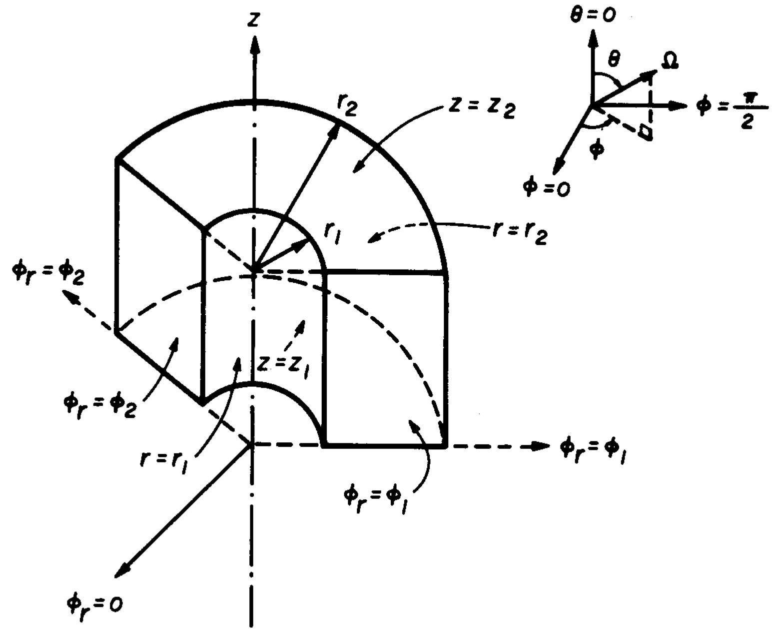

Fig. 3. The same as in Fig. 2 for a cylindrical element.

If we assume the cylindrical element to be a cylinder confined by $r = r_1$, $z = z_1$ and $z = z_2'$ only, an analytical solution is readily available. The boundary conditions are revised as follows:

$$\left.\begin{aligned}
&I_0{}^0 = \text{finite} && \text{at } r = r_1,\\
&\frac{\partial I_0{}^0}{\partial r} + hI_0{}^0 = 0 && \text{at } r = r_2,\\
&I_0{}^0(\phi_r) = I_0{}^0(\phi_r + 2\pi) && \text{at any } \phi_r,\\
&\left.\frac{\partial I_0{}^0}{\partial \phi_r}\right|_{\phi_r} = \left.\frac{\partial I_0{}^0}{\partial \phi_r}\right|_{\phi_r+2\pi}, && \\
&\frac{\partial I_0{}^0}{\partial z} \mp hI_0{}^0 = 0 && \text{at } \begin{cases} z = z_1(-),\\ z = z_2(+).\end{cases}
\end{aligned}\right\} \tag{4.20}$$

This time, the third and fourth equations satisfy the orthogonal requirement of the Sturm–Liouville problem. Thus, by the standard method of separation of variables, we can derive the following solutions:

$$I_0{}^0(r, \phi_r, z) = \sum_m (C_+ \, e^{\xi_m z} + C_- \, e^{-\xi_m z} + K_m) J_0(\zeta_m r), \tag{4.21}$$

where

$$\xi_m = \Lambda^2 + \zeta_m{}^2,$$

$$K_m = \frac{\Lambda^2 B_\nu(T)}{\xi_m{}^2 g_m} \int_0^{r_2} \frac{1}{r} J_0(\zeta_m r) \, dr,$$

$$g_m = \int_0^{r_2} \frac{1}{r} J_0(\zeta_m r) J_0(\zeta_m r) \, dr,$$

and J denotes the Bessel Function.

Moreover, ζ_m can be found from the equation

$$hJ_0(\zeta_m r_2) - \zeta_m J_1(\zeta_m r_2) = 0.$$

Equation (4.21) does not contain the ϕ_r-dependent term because the model considered is cylindrically symmetric.

(c) *Spherical coordinates*

In the case of spherical coordinates, Eq. (4.9) takes the form

$$\frac{1}{\rho^2}\frac{1}{\partial\rho}\rho^2\frac{\partial I_0^0}{\partial\rho} + \frac{1}{\rho^2\sin\theta_\rho}\frac{\partial}{\partial\theta_\rho}\left(\sin\theta_\rho\frac{\partial I_0^0}{\partial\theta_\rho}\right) + \frac{1}{\rho^2\sin^2\theta_\rho}\frac{\partial^2 I_0^0}{\partial\phi_\rho^2} - \Lambda^2 I_0^0$$
$$= \begin{cases} -\dfrac{3\sigma^2\tilde{\omega}F}{4}(\nu_1+\tilde{\omega}_1)\exp\{-\sigma[(\rho_0^2-\rho^2\sin^2\theta_\rho)^{1/2}-\rho\cos\theta_\rho]\} & \text{SOL,} \\ -\Lambda^2 B_\nu(T) & \text{IR,} \end{cases} \tag{4.22}$$

where the exponential term of the source function in the solar case is derived based on the assumption that the solar flux is in the direction of $\theta_\rho = 0$, where ρ is the radius of the sphere.

Because of its complexity, Eq. (4.22) has not been used to its full extent to study the transfer processes within a sphere. Simplifications, however, have been made using the spherically symmetric property to solve the equation. The present spherical harmonic method provides a systematic approach in reducing the basic equation to a form such that either an analytic or a numerical solution may be possible.

As shown in Fig. 4, for a spherical element bounded by six surfaces ($\rho = \rho_1$, $\rho = \rho_2$, $\phi_\rho = \phi_1$, $\phi_\rho = \phi_2$, $\theta_\rho = \theta_1$, and $\theta_\rho = \phi_2$) and governed by Eq. (4.22), derivation of boundary conditions involves the process of transformation of angular coordinates. We denote a new angular system (θ', ϕ') such that θ' is the angle between $\mathbf{\Omega}$ and $\boldsymbol{\rho}$, and ϕ' is the azimuthal angle on the plane perpendicular to $\boldsymbol{\rho}$, where $\boldsymbol{\rho}$ is the position vector. Based on these definitions, we find

$$\left.\begin{aligned} \Omega_\rho &= \cos\theta', \\ \Omega_{\theta_\rho} &= \sin\theta'\cos\phi', \\ \Omega_{\phi_\rho} &= \sin\theta'\sin\phi'. \end{aligned}\right\} \tag{4.23}$$

In order to evaluate the domain of integration in Eq. (4.12), a corresponding coordinate change is added to the spherical harmonic functions. We start with Eq. (3.3), and rewrite it for $N = 1$,

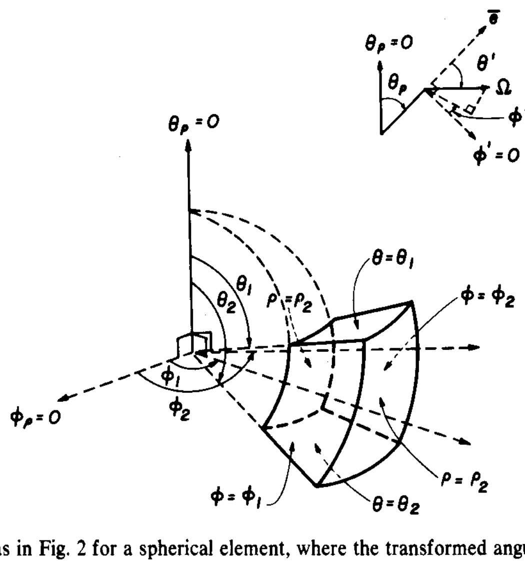

Fig. 4. The same as in Fig. 2 for a spherical element, where the transformed angular notation is shown.

i.e.,

$$I = I_0^0 + I_1^0\Omega_z + \frac{\sqrt{2}}{2}(I_1^{-1} - I_1^1)\Omega_x - \frac{\sqrt{2}}{2}i(I_1^{-1} + I_1^1)\Omega_y. \tag{4.24}$$

Furthermore, by using Eq. (2.9), we may express Eq. (4.24) in terms of (Ω_ρ, Ω_{θ_ρ}, and Ω_{ϕ_ρ}) in the form

$$I = I_0^0\left[I_1^0\frac{\sqrt{2}}{2}(I_1^{-1} - I_1^1)\frac{\sqrt{2}}{2}i(I_1^{-1} + I_1^1)\right]\boldsymbol{T}^{-1}_{\text{car-sph}}\begin{bmatrix}\Omega_\rho\\ \Omega_{\theta_\rho}\\ \Omega_{\theta_\rho}\end{bmatrix}. \tag{4.25}$$

Consequently, Eq. (4.11) may be modified to give

$$\int\int_{\Omega'} Y_l^m(\boldsymbol{\Omega}')I_{\text{in}}(\boldsymbol{\Omega}')\,d\Omega' = 0 \qquad \text{for } l = \text{odd}, \tag{4.26}$$

where Ω' is the new angular system (θ', ϕ'). The advantage in such a transformation is that the domain of integration can be defined easily. Table 3 lists the domain of integration based on the six bounding surfaces which are expressed as $\rho = \rho_1$ ($\rho_1 = 0$ in general), $\rho = \rho_2$, $\phi = \phi_1$, $\phi_\rho = \phi_2$, $\theta_\rho = \theta_1$, and $\theta_\rho = \theta_2$. Following similar analyses as those illustrated in previous sections, we find six equations at bounding surfaces in the forms

$$I_0^0 \pm \frac{2}{3}\left[\frac{1}{\sqrt{2}}(I_1^{-1} - I_1^1)\cos\phi_\rho \sin\theta_\rho - \frac{i}{\sqrt{2}}(I_1^{-1} + I_1^1)\sin\phi_\rho \sin\theta_\rho + I_1^0\cos\theta_\rho\right] = 0$$

$$\text{at } \rho = \rho_1(+) \text{ and } \rho = \rho_2(-),$$

$$I_0^0 \pm \frac{2}{3}\left[-\frac{1}{\sqrt{2}}(I_1^{-1} - I_1^1)\cos\theta_\rho \cos\phi_\rho - \frac{i}{\sqrt{2}}(I_1^{-1} + I_1^1)\cos\theta_\rho \sin\phi_\rho + I_1^0\sin\theta_\rho\right] = 0$$

$$\text{at } \theta_\rho = \theta_1(+) \text{ and } \theta_\rho = \theta_2(-),$$

$$I_0^0 \pm \frac{2}{3}\left[-\frac{1}{\sqrt{2}}(I_1^{-1} - I_1^1)\sin\phi_\rho + \frac{i}{\sqrt{2}}(I_1^{-1} + I_1^1)\cos\phi_\rho\right] = 0$$

$$\text{at } \phi_\rho = \phi_1(+) \text{ and } \phi_\rho = \phi_2(-). \tag{4.27}$$

Table 3. The domain of integration in spherical coordinates.

Surface	θ'	ϕ'
$\rho=\rho_1$	$0 - \frac{\pi}{2}$	$0 - 2\pi$
$\rho=\rho_2$	$\frac{\pi}{2} - \pi$	$0 - 2\pi$
$\phi_\rho=\phi_1$	$0 - \pi$	$0 - \pi$
$\phi_\rho=\phi_2$	$0 - \pi$	$\pi - 2\pi$
$\theta_\rho=\theta_1$	$0 - \pi$	$(-\frac{\pi}{2}) - \frac{\pi}{2}$
$\theta_\rho=\theta_2$	$0 - \pi$	$\frac{\pi}{2} - \frac{3\pi}{2}$

Again, we consider only the thermal i.r. case where the six equations can be further reduced to a set of homogeneous equations in terms of I_0^0 only. Thus we find

$$\frac{\partial I_0^0}{\partial \rho} \pm hI_0^0 = 0 \qquad \text{at } \rho = \rho_1(-) \text{ and } \rho = \rho_2(+),$$

$$\frac{\partial I_0^0}{\rho \partial \theta_\rho} \pm hI_0^0 = 0 \qquad \text{at } \theta_\rho = \theta_1(-) \text{ and } \theta_\rho = \theta_2(+), \tag{4.28}$$

$$\frac{\partial I_0^0}{\rho \sin \theta_\rho \partial \phi_\rho} \pm hI_0^0 = 0 \qquad \text{at } \phi_\rho = \phi_1(-) \text{ and } \phi_\rho = \phi_2(+).$$

Since the second and third sets of boundary conditions do not satisfy the requirements for a Sturm–Liouville problem, an analytical solution at this point is impossible. One exception is when the model is assumed to be a spherical ball. When the ball is bounded by the surface $\rho = \rho_2$, Eq. (4.20) reduces to the one-dimensional form. With the assumption of spherical symmetry, only the first set of boundary conditions is necessary. The solution in this case will be discussed in the next section along with other one-dimensional models of different coordinate systems.

5. APPLICATION OF THE SOLUTION

As an example of the application of the solutions derived in previous sections, we shall show the variation of the emissivity on the boundary of three model clouds, i.e., the plane-parallel, the axially infinite cylinder and the sphere. In each case, the model cloud is assumed to be in a vacuum so that there is no incident energy from outside the boundary. Therefore, only the cloud emission contributes to the source function. The interior composition of the cloud is assumed to be homogeneous. Next, we define the emissivity in the form

$$\epsilon = F_{out}/[\pi B_\nu(T_c)], \tag{5.1}$$

where T_c is the cloud temperature and F_{out} is the local outward flux density normal to a differential surface element around the point. As is evident in Eq. (5.1), the cloud boundary will approach the behavior of a black surface if the emissivity is very close to 1.

Table 4 lists the governing equations and boundary conditions as derived from the three-dimensional equations described in previous sections. The solution for I_0^0 is readily available and it is listed in Table 5. In a straightforward manner, the emissivity can be found by

Table 4. The governing equations and boundary conditions for the three model clouds considered in Section 5.

Model	Governing Equation	Boundary Condition	
Plane-Parallel	$\frac{d^2 I_0^0}{dz^2} - \Lambda^2 I_0^0 = -\Lambda^2 B_\nu(T)$	$\frac{dI_0^0}{dz} \pm hI_0^0 = 0$	at $z=0(-)$, $z=z_0(+)$
Cylindrical	$\frac{1}{r}\frac{d}{dr} r \frac{dI_0^0}{dr} - \Lambda^2 I_0^0 = -\Lambda^2 B_\nu(T)$	$I_0^0 = \text{finite}$	at $r=0$
		$\frac{dI_0^0}{dr} + hI_0^0 = 0$	at $r=r_0$
Spherical	$\frac{1}{\rho^2}\frac{d}{d\rho} \rho^2 \frac{dI_0^0}{d\rho} - \Lambda^2 I_0^0 = -\Lambda^2 B_\nu(T)$	$I_0^0 = \text{finite}$	at $\rho=0$
		$\frac{dI_0^0}{d\rho} + hI_0^0 = 0$	at $\rho=\rho_0$

Table 5. The solutions for I_0^0 for the three model clouds considered in Section 5, $\xi = \Lambda/3\sigma\gamma_1$; I_0, I_1 = modified Bessel functions of the first kind; i_0, i_1 = modified spherical Bessel functions of the first kind.

Model	$I_0^0/B_\nu(T_u)$
Plane-Parallel	$1 - \dfrac{e^{\Lambda z} + e^{\Lambda(z_0 - z)}}{(e^{\Lambda z_0}+1) + 2\xi\,(e^{\Lambda z_0}-1)}$
Cylindrical	$1 - \dfrac{I_0^0(\Lambda r)}{I_0(\Lambda r_0) + 2\xi I_1(\Lambda r_0)}$
Spherical	$1 - \dfrac{i_0(\Lambda\rho)}{i_0(\Lambda\rho_0) + 2\xi i_1(\Lambda\rho_0)}$

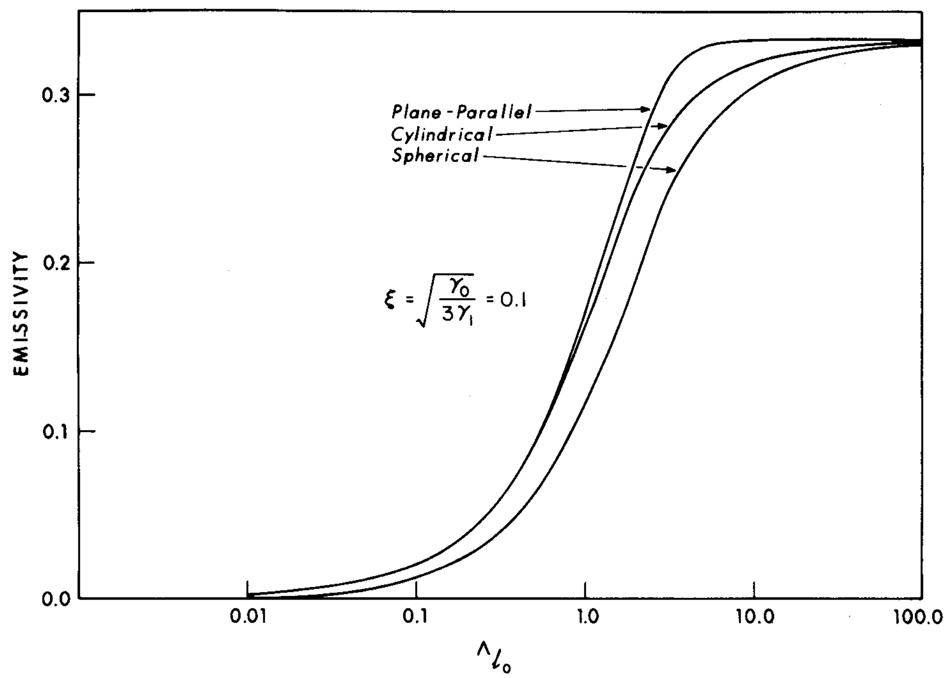

Fig. 5. Emissivity vs Λl_0 for $\xi = 0.1$, as calculated based on equations listed in Table 5.

expressing F_{out} in terms of I_0^0 at the boundary as

$$F_{out} = \pi I_0^0 + \frac{2\pi}{3\sigma\gamma_1}\frac{dI_1^0}{dl^*}, \tag{5.2}$$

where $l^* = z$ in the plane-parallel case, $l^* = r$ in the cylindrical case and $l^* = \rho$ in the spherical case. Upon using the boundary conditions, we find

$$F_{out} = 2\pi I_0^0 \tag{5.3}$$

and it follows that at the boundary

$$\epsilon = 2\pi I_0^0/B_\nu(T_c). \tag{5.4}$$

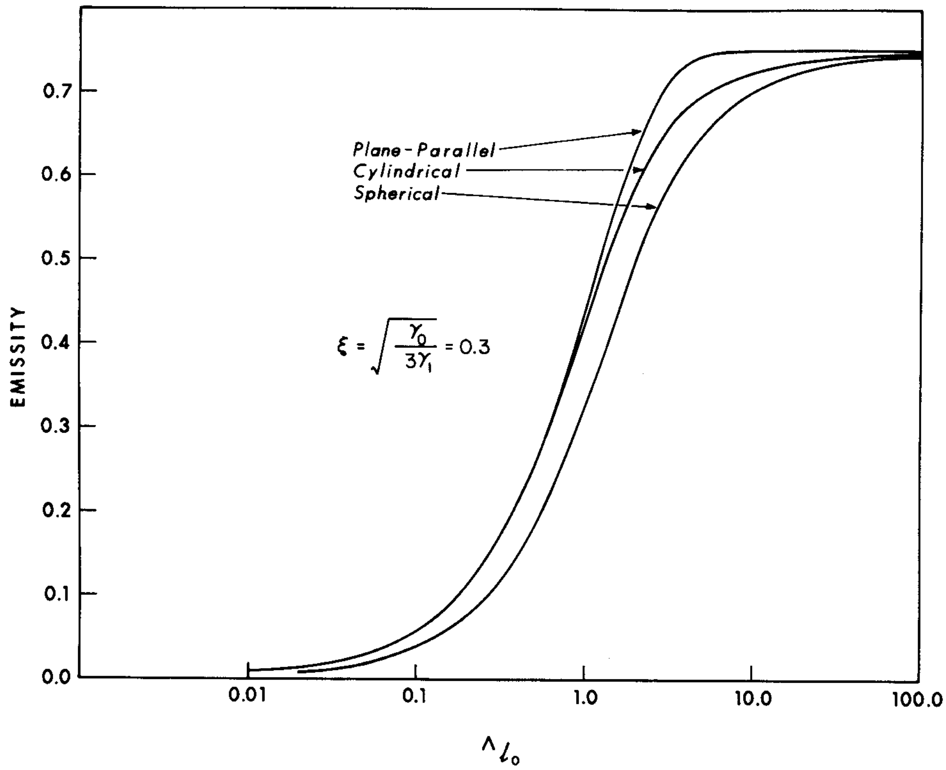

Fig. 6. The same as in Fig. 5 for $\xi = 0.3$.

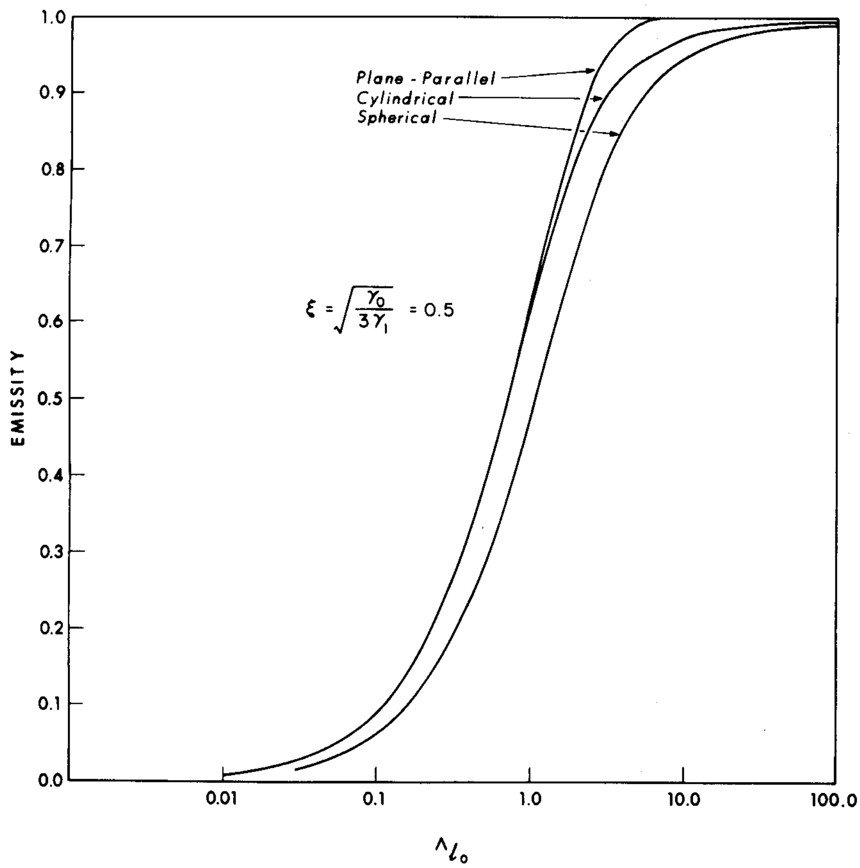

Fig. 7. The same as in Fig. 5 for $\xi = 0.5$.

It should be noted that the emissivity is a function of two variables Λl_0 and ξ, so that we may plot ϵ vs Λl_0 with ξ as a parameter. This is shown in Figs. 5–7 for $\xi = 0.1$, 0.3 and 0.5, respectively. The values of Λl_0 may vary from zero to infinity, but those of ξ are only between 0 and $\sqrt{1/3}$. From these figures, we notice that the asymptotic behavior of ϵ is dependent on ζ. Thus, for $\xi = 0.1$, $\epsilon(\infty) \sim 0.37$ and for $\xi = 0.5$, $\epsilon(\infty) \sim 1.0$. It is expected that $\epsilon(\infty)$ may slightly exceed unity as ξ approaches its upper limit because of the approximate nature of the first-order expansion. In addition, the cloud shape produces some differences in the emissivity value for a given value of Λl_0. In general, a spherical cloud has the lowest emissivity value, because it contains less emitting media than the other two models. This trend was also noted by Liou and Ou[8] when a comparison between a plane-parallel model and a rectangular model was made.

6. CONCLUSION

In this study, we have applied and generalized the spherical harmonic method to solve the basic radiative transfer equation by means of the harmonics decomposition in such a way that a set of partial differential equations are derived for the three fundamental coordinate systems. The first-order approximation is then used to truncate the number of equations into four. After a number of mathematical analyses, a modified Helmholtz equation is obtained. All possible analytical solutions for the modified Helmholtz equations in Cartesian, cylindrical, and spherical coordinates with appropriate boundary conditions are subsequently derived in forms of series expansion. Numerical computations are finally carried out for flux emissivities for the three coordinate systems in one-dimensional space. Results show that the emissivity from a spherical cloud is lower than that from a cylindrical or a plane-parallel cloud.

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