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Applications of the Discrete-Ordinate Method for Radiative Transfer to Inhomogeneous Aerosol Atmospheres

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The discrete-ordinate method for radiative transfer is applied to an inhomogeneous atmosphere containing molecules and aerosols. The unknown coefficients in the analytic solution to the transfer equation are determined from boundary conditions of the diffuse intensity at the top and bottom of the atmosphere and from continuity conditions of radiation at the interface of the predivided homogeneous layers. The assumption of the homogeneity of the atmosphere is shown to overestimate the reflection (local albedo) and to underestimate the diffuse transmission at the bottom of the atmosphere. On the basis of calculations for the transfer of solar radiation in inhomogeneous hazy atmospheres we also show that the increase or decrease of the local albedo, due to the additional load of aerosols in the atmospheric boundary layer, depends on the characteristics of the surface albedo.

1. INTRODUCTION

The discrete-ordinate method for radiative transfer was introduced originally by *Chandrasekhar* [1950]. It has been subsequently developed by *Piotrowski* [1956], *Lenoble* [1956], *Keller* [1958], *Samuelson* [1967], and recently by *Yamamoto et al.* [1971] and *Liou* [1973] with applications to cloudy and hazy atmospheres.

The primary advantages of the discrete-ordinate method may be summarized as follows:

1. The solution of the transfer equation may be derived explicitly and, consequently, the intensity and flux calculations are independent of the total optical depth of cloud and aerosol layers.
2. The method yields the internal radiation field as well as the reflection and transmission without the additional computation effort.
3. Analytic two-stream and four-stream solutions can be derived in closed forms [*Liou*, 1974] which are particularly useful for calculating the radiation fluxes in the atmosphere. (The two-stream approximation for radiative transfer introduces large errors for optically thin cases. In a recent paper, *Coakley and Chýlek* [1975] developed a modified two-stream technique which seems to be fairly accurate for optically thin atmospheres.) The computer time requirement for these two approximations is relatively small as compared with other transfer techniques.

The discrete-ordinate method described above applies to an atmosphere which is vertically homogeneous with respect to the concentration of molecules and/or particles. In such an atmosphere a nondimensional optical depth is employed to describe the optical property of the entire atmosphere. However, in realistic cloudy and hazy atmospheres the concentrations of cloud and aerosol particles, as well as molecules, normally vary with height, so that the assumption of the vertical homogeneity may not be valid. In this paper we explore the applicability of the discrete-ordinate method for radiative transfer to inhomogeneous atmospheres.

In section 2 we discuss analytic solutions for homogeneous layers on the basis of the discrete-ordinate method. In section 3 the boundary and continuity equations for the diffuse intensity are derived, so that a complete set of linear equations can be employed to determine the unknown constants of proportionality in the analytic solutions. Some computational problems associated with an atmosphere containing aerosols and molecules are further discussed in section 4. Finally, results of the reflected and transmitted fluxes for aerosol atmospheres are presented, and some comparisons with those obtained from the assumption of the vertical homogeneity are carried out.

2. SOLUTIONS OF THE TRANSFER EQUATION FOR HOMOGENEOUS LAYERS

We first divide the entire atmosphere into N layers (Figure 1) according to the height Z. Within each layer the atmosphere is considered to be homogeneous with respect to the single-scattering albedo $\tilde{\omega}_0$ and the phase function $P(\Theta)$, where Θ is the scattering angle.

For the simplicity of discussion in this paper we shall neglect the azimuthal dependence of the scattered intensity (or radiance) and consider the following transfer equation for a plane-parallel homogeneous layer

$$\mu \frac{dI^l(\tau, \mu)}{d\tau} = I^l(\tau, \mu) - \frac{1}{2} \int_{-1}^{+1} P^l(\mu, \mu') I^l(\tau, \mu')\, d\mu' - \tfrac{1}{4} F_0 P^l(\mu, \mu_0) \exp(-\tau/\mu_0) \quad (1)$$

where I represents the intensity, τ the optical depth, πF_0 the incident solar flux, and μ and μ_0 the cosine of the emergent and solar zenith angles, respectively.

Following the procedures described by *Liou* [1973], the analytic solutions for (1) based upon the discrete-ordinate method for radiative transfer are given by

$$I^l(\tau, \mu_i) = \sum_j L_j{}^l \phi_j{}^l(\mu_i) \exp(-k_j{}^l \tau) + Z^l(\mu_i) \exp(-\tau/\mu_0) \qquad i = -n, \cdots, n \quad (2)$$

in (2), $\sum_j$ denotes j from $-n$ to n ($n \neq 0$), and the eigenfunctions are derived from the associated homogeneous system as

$$\phi_j{}^l(\mu_i) = \frac{1}{1 + \mu_i k_j{}^l} \sum_{m=0}^{M} \tilde{\omega}_m{}^l \xi_m(k_j{}^l) p_m{}^l(\mu_i) \quad (3)$$

where M denotes the number of terms in the Legendre polynomial, $p_m{}^l$, expansion, $\tilde{\omega}_m{}^l$ is a set of $M + 1$ constants, and ξ_m is a constant of proportionality and can be determined by

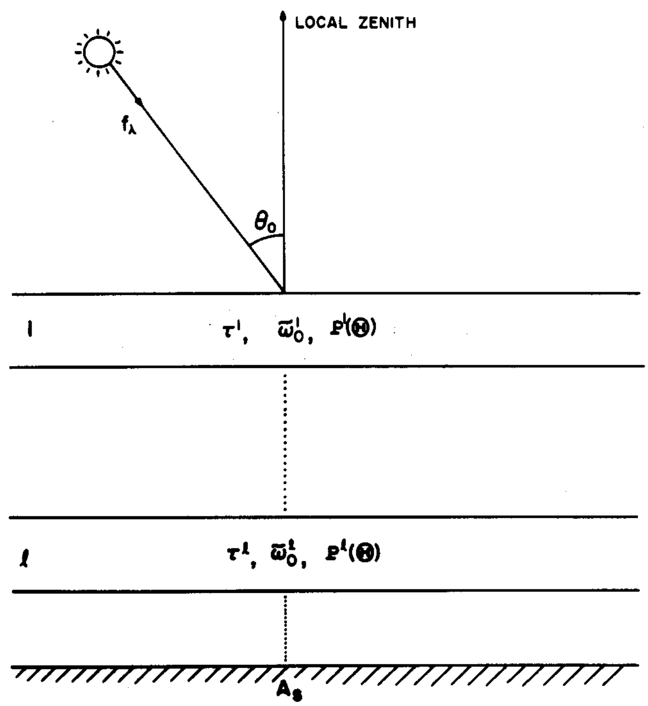

Fig. 1. The inhomogeneous atmosphere is divided into N homogeneous layers with respect to the single-scattering albedo and phase function. Here f_λ $(= \pi F_0)$ denotes the solar flux at the top of the atmosphere.

$$\xi_{m+1} = \frac{2m + 1 - \tilde{\omega}_m{}^l}{k^l(m + 1)} \xi_m - \frac{m}{m + 1} \xi_{m-1} \qquad m = 0, 1, \cdots, M - 1 \tag{4}$$

By taking $\xi_0 = 1$, all values of ξ can be evaluated. The eigenvalue k^l may also be determined from

$$f(k^l) = 1 - \tfrac{1}{2} \sum_i a_i \phi_i{}^l(\mu_i) = 0 \tag{5}$$

The last term in (2) represents the particular solution where

$$Z^l(\mu_i) = \frac{1}{4}\mu_0 F_0 \frac{H^l(\mu_0) H^l(-\mu_0)}{\mu_0 + \mu_i} \sum_{m=0}^{M} \tilde{\omega}_m{}^l \xi_m \left(\frac{1}{\mu_0}\right) p_m{}^l(\mu_i) \tag{6}$$

and the H function [*Chandrasekhar,* 1950]

$$H^l(\mu) = \frac{1}{\mu_1 \mu_2 \cdots \mu_n} \prod_{i=1}^{n} (\mu + \mu_i) \left[\prod_{j=1}^{n} (1 + k_j{}^l \mu) \right]^{-1} \tag{7}$$

Note that in the above equations the superscript l denotes the value for a given layer. Finally, $L_j{}^l$ are unknown constants of proportionality to be determined from the boundary and continuity equations discussed in the following section.

The solution expressed in (2) is valid only for nonconservative scattering. For conservative scattering a slightly different form can easily be derived, but we neglect the solution here for simplicity.

3. BOUNDARY AND CONTINUITY EQUATIONS

In reference to Figure 1, at the top of the atmosphere there is no diffuse downward intensity, so that

$$I^1(0, -\mu_i) = 0 \qquad i = 1, 2, \cdots, n \tag{8}$$

Within the atmosphere the upward and downward intensities have to be continuous at the interface of each homogeneous atmosphere. Thus we shall have

$$I^l(\tau^l, \mu_i) = I^{l+1}(\tau^l, \mu_i) \qquad i = -n, \cdots, n \qquad l = 1, 2, \cdots, N - 1 \tag{9}$$

Note that τ^l represents the optical depth from the top of the atmosphere to the bottom of the l layer, and $\mu_{-i} = -\mu_i$. The above relations have been previously noted by *Shettle and Weinman* [1970], *Weinman and Guetter* [1972], and *Hunt and Grant* [1969].

At the bottom of the atmosphere, assuming an isotropic reflection surface with an albedo A_s, the upward diffuse intensity can be expressed in terms of the downward diffuse flux plus the downward direct flux as

$$I^N(\tau^N, +\mu_i) = \frac{A_s}{\pi} [F^\downarrow(\tau^N) + \mu_0 \pi F_0 \exp(-\tau^N/\mu_0)] \qquad i = 1, 2, \cdots, n \tag{10}$$

where

$$F^\downarrow(\tau^N) = 2\pi \sum_{i=1}^{n} I^N(\tau^N, -\mu_i) a_i \mu_i \tag{11}$$

with a_i being the Gauss quadrature weight.

Substituting (2) into (8)–(10), we obtain the following set of equations for the determination of $L_j{}^l$:

$$\sum_j L_j{}^1 \phi_j{}^1(-\mu_i) = -Z^1(-\mu_i) \qquad i = 1, 2, \cdots, n \tag{12}$$

$$\sum_j [L_j{}^l \gamma_j{}^l(\mu_i) + L_j{}^{l+1} \delta_j{}^{l+1}(\mu_i)] = -{}^l\eta^{l+1}(\mu_i) \qquad i = -n, \cdots, -1, 1, \cdots, n \qquad l = 1, 2, \cdots, N - 1 \tag{13}$$

$$\sum_j L_j{}^N \beta_j{}^N(+\mu_i) = -\epsilon^N(+\mu_i) \qquad i = 1, 2, \cdots, n \tag{14}$$

where

$$\gamma_j{}^l(\mu_i) = \phi_j{}^l(\mu_i) \exp(-k_j{}^l \tau^l) \tag{15}$$

$$\delta_j{}^{l+1}(\mu_i) = -\phi_j{}^{l+1}(\mu_i) \exp(-k_j{}^{l+1} \tau^l) \tag{16}$$

$$\beta_j{}^N(+\mu_i) = \left[\phi_j{}^N(+\mu_i) - 2A_s \sum_{i=1}^{n} \phi_j{}^N(-\mu_i) a_i \mu_i \right] \times \exp(-k_j{}^N \tau^N) \tag{17}$$

$${}^l\eta^{l+1}(\mu_i) = [Z^l(\mu_i) - Z^{l+1}(\mu_i)] \exp(-\tau^l/\mu_0) \tag{18}$$

$$\epsilon^N(+\mu_i) = \left[Z^N(+\mu_i) - 2A_s \sum_{i=1}^{n} Z^N(-\mu_i) a_i \mu_i - \frac{A_s}{\pi} \mu_0 \pi F_0 \right] \exp(-\tau^N/\mu_0) \tag{19}$$

Hence we have $N \times 2n$ equations for the determination of $N \times 2n$ unknown constants $L_j{}^l$. In terms of matrix operations we may write

$$\mathbf{\Phi L} = \boldsymbol{\chi} \tag{20}$$

The coefficient matrix in (20) is

$$\mathbf{L} = \begin{bmatrix} L_{-n}^{1} \\ \vdots \\ L_{n}^{1} \\ L_{-n}^{2} \\ \vdots \\ L_{n}^{2} \\ \vdots \\ L_{-n}^{N} \\ \vdots \\ L_{n}^{N} \end{bmatrix} \qquad (21)$$

The matrix denoting the contribution due to solar flux may be written as

$$\chi = -\begin{bmatrix} Z^{1}(-\mu_n) \\ \vdots \\ Z^{1}(-\mu_1) \\ {}^{1}\eta^{2}(-\mu_n) \\ \vdots \\ {}^{1}\eta^{2}(+\mu_n) \\ \vdots \\ \epsilon^{N}(+\mu_1) \\ \vdots \\ \epsilon^{N}(+\mu_n) \end{bmatrix} \qquad (22)$$

And finally, the $N \times 2n$ by $N \times 2n$ matrix

$$\mathbf{\Phi} = \begin{bmatrix} \phi_{-n}^{1}(-\mu_n) \cdots \phi_{n}^{1}(-\mu_n) & & & \\ \vdots \qquad \vdots & & & \\ \phi_{-n}^{1}(-\mu_1) \cdots \phi_{n}^{1}(-\mu_1) & & & \\ \gamma_{-n}^{1}(-\mu_n) \cdots \gamma_{n}^{1}(-\mu_n) & \delta_{-n}^{2}(-\mu_n) \cdots \delta_{n}^{2}(-\mu_n) & & \\ \vdots \qquad \vdots & \vdots \qquad \vdots & & \\ \gamma_{-n}^{1}(\mu_n) \cdots \gamma_{n}^{1}(\mu_n) & \delta_{-n}^{2}(\mu_n) \cdots \delta_{n}^{2}(\mu_n) & & \\ & \vdots \qquad \vdots & \vdots \qquad \vdots & \\ & & \beta_{-n}^{N}(\mu_1) \cdots \beta_{n}^{N}(\mu_1) & \\ & & \vdots \qquad \vdots & \\ & & \beta_{-n}^{N}(\mu_n) \cdots \beta_{n}^{N}(\mu_n) & \end{bmatrix} \qquad (23)$$

where the blank spaces denote zero elements.

With all the above information we may now carry out the computations for the evaluation of L_j^l simultaneously by means of any matrix inversion method. Then we may insert values of L_j^l into (2) to obtain the intensity distribution within each layer.

4. Computational Aspects and Results

In an atmosphere containing molecules and aerosols the single-scattering albedo and the normalized phase function for the layer l may be written as

$$\tilde{\omega}_0^{\,l} = \frac{\Delta\tau_R + \Delta\tau_{M,s}}{\Delta\tau_R + \Delta\tau_{M,s} + \Delta\tau_{M,a}} \qquad (24)$$

$$P^l(\Theta) = \frac{\Delta\tau_R P_R(\Theta) + \Delta\tau_{M,s} P_M(\Theta)}{\Delta\tau_R + \Delta\tau_{M,s}} \qquad (25)$$

respectively, where $P_R(\Theta)$ and $P_M(\Theta)$ are phase functions for a volume of molecules and aerosols, respectively, $\Delta\tau_R$ denotes the Rayleigh scattering optical depth, and $\Delta\tau_{M,s}$ and $\Delta\tau_{M,a}$ represent the optical depths due to the scattering and absorption of aerosol particles, respectively. All the above quantities are for a monochromatic wavelength. Since the phase function and scattering cross section for Rayleigh molecules are well known, we shall make no further discussion about them. The phase function and optical depths for a volume of aerosol particles are evaluated on the basis of the following assumptions:

1. Aerosols are spherical homogeneous particles.
2. The particle size distribution of aerosols $n(r)$ in the atmosphere may be described by

$$\begin{aligned} \frac{dn(r)}{dr} &= c_1 r^{-4} && 0.1 < r \leq 10\ \mu\text{m} \\ \frac{dn(r)}{dr} &= c_1 10^{4} && 0.02 \leq r < 0.1\ \mu\text{m} \\ \frac{dn(r)}{dr} &= 0 && \text{otherwise} \end{aligned} \qquad (26)$$

where r denotes the particle radius and the constant $c_1 = 0.883 \times 10^{-3}$.

3. The real and imaginary parts of the refractive index are taken to be 1.5 and 0.00714, respectively, for a visible wavelength of 0.7 μm.

With the hypothetical information of the shape, the size, and the refractive index of aerosol particles we may then carry out single-scattering computations employing the well-known Mie scattering theory [e.g., *Liou and Hansen*, 1971] to obtain the phase function and the scattering and absorption cross sections. Furthermore, with the additional information on the

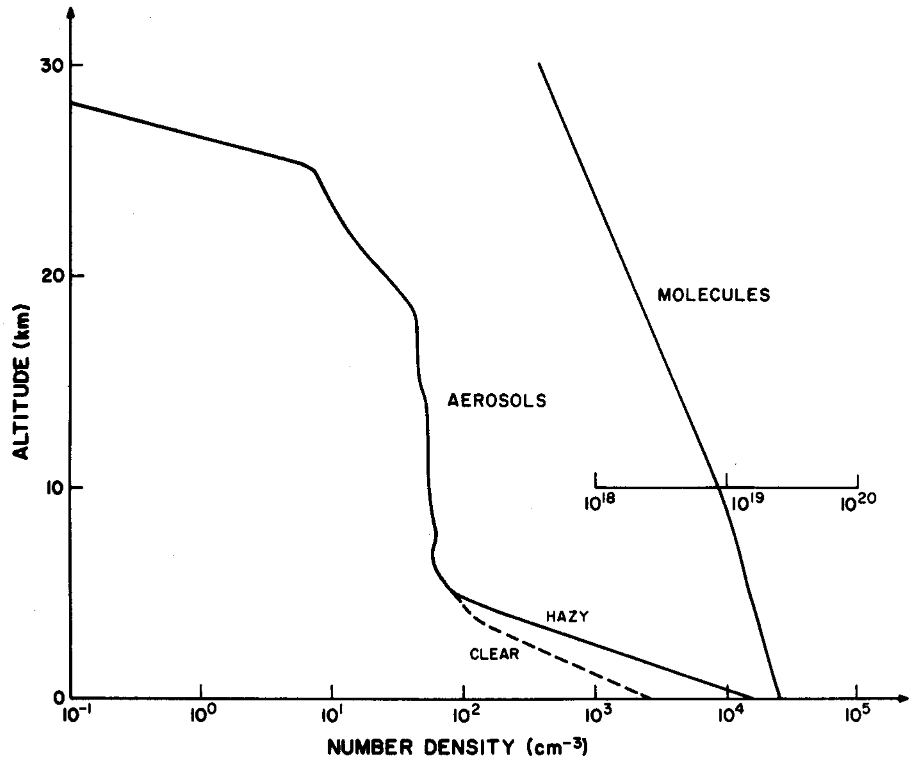

Fig. 2. The number density of aerosols and molecules as functions of height in model atmospheres. Two aerosol concentrations are shown.

vertical distribution of aerosol and molecular concentrations (Figure 2, after *McClatchey et al.* [1971]) the scattering and absorption optical depths for a layer l may be evaluated.

In the discrete-ordinate method for radiative transfer the phase function for the azimuthal independent case is expanded in Legendre polynomials p_m as follows:

$$P^l(\mu, \mu') = \sum_{m=0}^{M} \tilde{\omega}_m{}^l p_m(\mu) p_m(\mu') \qquad (27)$$

where $\tilde{\omega}_m{}^l$ is a set of $M + 1$ constants that can be determined by

$$\tilde{\omega}_m{}^l = \frac{2m+1}{2} \int_{-1}^{+1} P^l(\cos\Theta)\, p_m(\cos\Theta)\, d\cos\Theta \qquad (28)$$

Note that for the convenience of discussion the phase functions in this section are all normalized such that their integrations over the solid angle become 1. Thus in (27) we shall have $\tilde{\omega}_0{}^l = 1$. Consequently, in order to insert the single-scattering properties of a mixture of aerosols and molecules into the transfer program, multiplication of the phase function in (27) by the single-scattering albedo in (24) is required.

Figure 3 shows the normalized phase functions for Mie aerosols and Rayleigh molecules. The former case is for a wavelength of 0.7 μm. It is clear that aerosols scatter much more radiation in forward directions at the expense of backward directions. The phase function for a volume of aerosols and molecules has to be evaluated according to (25).

In the transfer calculations the inhomogeneous atmosphere is divided into 10 layers (i.e., $N = 10$). For each layer, mean concentration values for aerosols and molecules are obtained by averaging the values at the top and bottom layers. Thus the phase function, the single-scattering albedo, and the total optical depth may be evaluated on the basis of the discussions

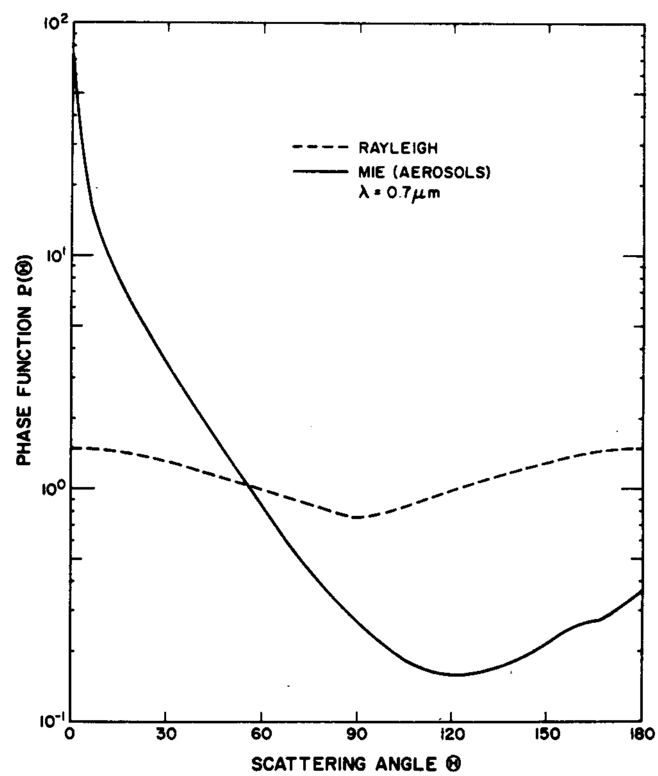

Fig. 3. The phase function of aerosols (solid line) for a visible wavelength of 0.7 μm calculated from the Mie scattering theory. The phase function (dotted line) of Rayleigh molecules is presented for comparison.

above. Then these values are incorporated into the radiative transfer program to obtain the reflected and transmitted intensity and flux. In this study, discrete streams of eight are employed in the calculations. We further define the reflection and diffuse transmission as

$$R = \frac{F^{\uparrow}(0)}{\mu_0 \pi F_0} \qquad T = \frac{F^{\downarrow}(\tau_N)}{\mu_0 \pi F_0} \tag{29}$$

respectively.

Figure 4 illustrates the reflection and diffuse transmission of the hazy atmosphere for the surface albedos of 0 and 0.4. The dotted lines represent results for an inhomogeneous atmosphere where heavy concentrations of aerosols are placed near the ground level. The solid lines in this figure denote those obtained from the assumption that the atmosphere is homogeneous in which single values of the phase function, the single-scattering albedo ($\tilde{\omega}_0$ = 0.934), and the total optical depth of the atmosphere (τ = 1.104) are derived. Comparisons between these two cases show that the assumption of the homogeneity of the atmosphere apparently overestimates values of the reflection and underestimates values of the diffuse transmission. The differences are much more pronounced when the surface albedo A_s = 0, because contributions of the scattered and absorbed radiation arise from the atmosphere alone. However, when a larger surface albedo of 0.4 is introduced, the effect of ground reflections compensates somewhat the errors produced by the homogeneity assumption. We note that smaller values of the reflection for the inhomogeneous hazy atmosphere are physically due, in part, to the fact that most of aerosols concentrate near the earth's surface. Other interesting results obtained from these calculations may be summarized as follows: First, increasing the surface albedo from 0 to 0.4 produces a large increase of the reflection from 0.12 to 0.35 when the sun is overhead. Second, the effect of the ground reflection is gradually reduced as the solar zenith angle increases ($\mu_0 \rightarrow 0$). Finally, an increase of the ground reflection also introduces larger values for the diffuse transmission.

Figure 5 illustrates the reflection and diffuse transmission of the clear atmosphere. The effect of the inhomogeneity appears more significant, particularly for A_s = 0. A difference by as much as 5% is seen for both the reflection and diffuse transmission for almost all solar zenith angles. The total

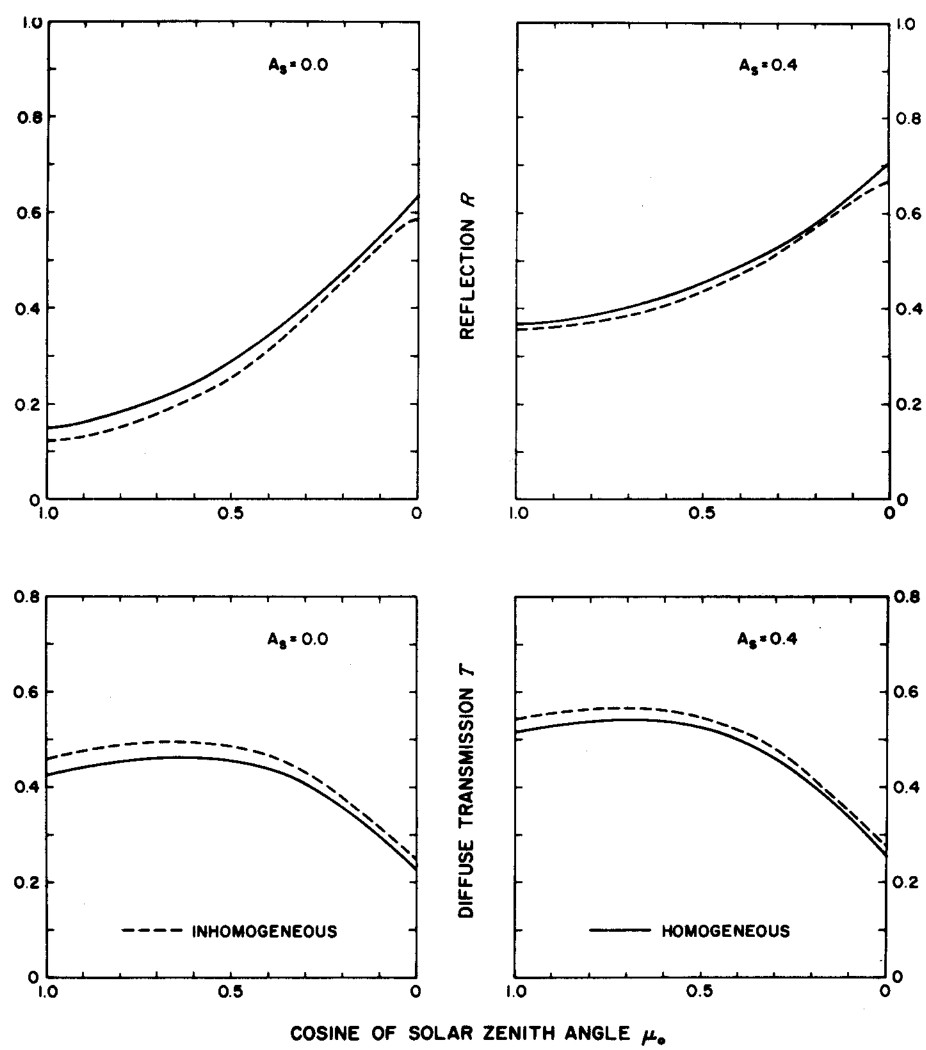

Fig. 4. The reflection and diffuse transmission (see text for definitions) as functions of the solar zenith angle for hazy inhomogeneous (dotted lines) and homogeneous (solid lines) atmospheres. Two surface albedos of 0.0 and 0.4 are employed in the calculations.

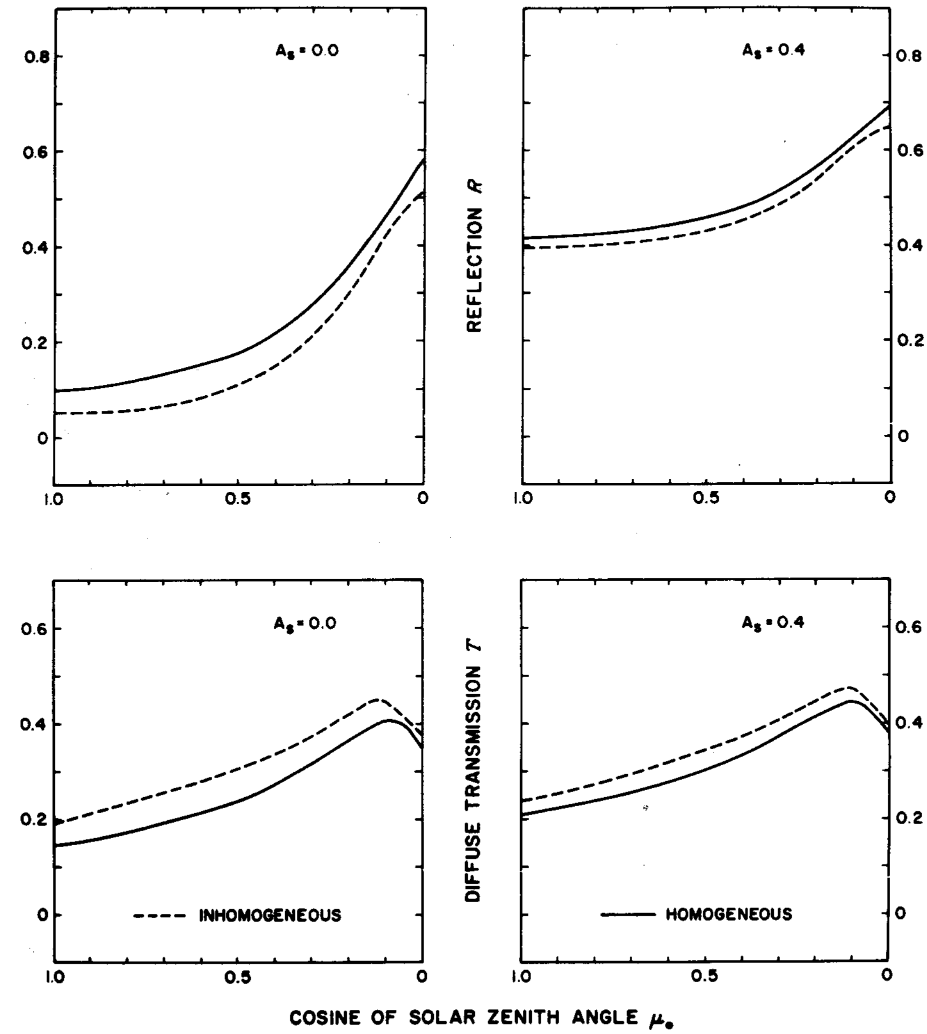

Fig. 5. Same as Figure 4 but for a clear atmosphere.

optical depth for the homogeneous clear atmosphere is 0.302 with a single-scattering albedo of 0.940. In this case the aerosol atmosphere is essentially optically thin, so that the underlying surface albedo plays a dominant role for the transfer of solar radiation in the atmosphere. Note that in the lower diagrams the direct transmission [= exp $(-\tau/\mu_0)$] of the solar radiation has not been included. This component represents important contributions to the downward flux within the atmosphere. The diffuse transmission for the inhomogeneous clear atmosphere seems to have a maximum value at μ_0 of about 0.2 regardless of the values of the surface albedo.

5. Conclusions

In this paper we have demonstrated the extension of the discrete-ordinate method for radiative transfer to an inhomogeneous atmosphere. The unknown coefficients in the analytic solution to the transfer equation are determined from boundary conditions for the diffuse intensity at the top and bottom of the atmosphere and by matching the diffuse intensity at the interface of the predivided homogeneous layers. Calculations of the reflected and transmitted intensity and flux based on the matrix formulation are carried out for hazy and clear atmospheres containing aerosols and molecules.

In view of the calculations illustrated in Figures 4 and 5 it appears that effects of the inhomogeneity of the atmosphere cannot be ignored, especially for low surface albedos. The assumption of the homogeneity of the atmosphere overestimates the reflection (local albedo) at the top of the atmosphere and underestimates the diffuse transmission at the bottom of the atmosphere.

The value of the local albedo of an aerosol atmosphere is of great interest to the study of the effect of man-made aerosols on the solar radiation and the possible implications of climatic changes. Assuming that the visible wavelength of 0.7 μm may be employed to denote the solar spectrum, we note from the upper graphs of Figures 4 and 5 that for $A_s = 0$ an increase of aerosol concentrations in the atmospheric boundary layer is to increase the local albedo by as much as 10% for most of the solar zenith angles. Consequently, it would cause a cooling effect for the earth-atmosphere system as a whole. On the contrary, however, the local albedo decreases by about 5% (μ_0 = 1) owing to the additional load of aerosols to a surface whose albedo is 0.4. Hence we would anticipate a slight warming for the earth-atmosphere system.

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