A Simple Formulation of the Delta-Four-Stream Approximation for Radiative Transfer Parameterizations

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ABSTRACT

We present a systematic development of the delta-four-stream approximation for calculations of radiative fluxes in planetary atmospheres. We illustrate that an analytic solution for this approximation can be derived explicitly, with minimum computational effort for flux calculations. Relative accuracy checks for reflection, transmission, and absorption for numerous asymmetry factors, single-scattering albedos, optical depths, and solar zenith angles have been performed with respect to the "exact" results computed from the adding method for radiative transfer. Overall, results from the delta-four-stream approximation yield relative accuracies within about 5%. This approximation is well suited to radiative transfer parameterizations involving flux and heating calculations in aerosol and cloudy atmospheres.

1. Introduction

Two-stream approximations for radiative transfer have been widely used in radiative flux calculations, as described in a summary paper by Meador and Weaver (1980). The popularity of two-stream approximations is because analytic solutions for upward and downward fluxes can be derived and numerical computations for these fluxes can be efficiently performed. The incorporation of the delta (δ)-function adjustment to account for the strong forward scattering of large particles in the context of two-stream approximations leads to a significant improvement in the accuracy of radiative flux calculations (see, e.g., Joseph et al. 1976). Basically, the δ-adjustment provides a third term closure through the second moment of the phase function expansion. King and Harshvardhan (1986) comprehensively examined the accuracy of various two-stream approximations. They indicated that relative errors of 15%–20% could be produced for a number of optical depths, solar zenith angles, and single-scattering albedos.

If the demand for accuracy in the flux calculations is high, say, on the order of 5%, it would be desirable to explore a methodology that is simple but more accurate. Along this line, Liou (1974) proposed that the four-stream approximation could be valuable for atmospheric flux computations. For this approximation, the solution for eigenvalues associated with the homogeneous part of the discretized equations can be derived analytically from the recurrence equation for eigenvalues. Thus, the computational time for flux calculations does not significantly exceed that for the two-stream approximation. Cuzzi et al. (1982), using the procedure suggested by Liou (1974), carried out an examination of the four-stream approximation. They illustrated that indeed the four-stream approximation, as well as the incorporation of the forward-peak adjustment in this approximation, has much to offer in flux calculations in terms of both accuracy and efficiency.

However, the four-stream approximation, as it is given in Liou (1974), is based on the general solution for the discrete-ordinates method for radiative transfer. To understand the merit of the four-stream approximation, a background in solving a set of differential equations based on Chandrasekhar's (1950) formulations is required. In particular, it is noted that the search for eigenvalues from the recurrence equation developed in the solution is both mathematically ambiguous and numerically troublesome (Liou 1973; Asano 1975; Stamnes and Swanson 1981).

This note on the four-stream approximation is intended to provide a systematic and independent development of the solution for this approximation. Specifically, this paper deals with the computation of solar radiative fluxes using a relatively simple, convenient, and accurate method. Knowledge of the discrete-or-

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dinates method for radiative transfer is desirable but not necessary. In addition, we provide a wide range of accuracy checks for this approximation, including the δ-adjustment to account for the forward diffraction peak based on the generalized similarity principle for radiative transfer. The methodology presented here should be useful for the calculation of solar fluxes in aerosol and cloudy atmospheres.

2. Discretization of the radiative transfer equation

Since we are concerned with flux calculations, we begin with the following basic equation governing the transfer of diffuse solar intensity I in plane-parallel atmospheres in the form

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{\tilde{\omega}}{2} \int_{-1}^{1} I(\tau, \mu') P(\mu, \mu') d\mu' - \frac{\tilde{\omega}}{4\pi} \pi F_0 P(\mu, -\mu_0) e^{-\tau/\mu_0}, \quad (2.1)$$

where $\mu = \cos\theta$, θ is the zenith angle, τ the normal optical depth, $\tilde{\omega}$ the single-scattering albedo, πF_0 the solar flux at the top of the atmosphere, and μ_0 the cosine of the solar zenith angle θ_0. The azimuth-independent phase function may be obtained from

$$P(\mu, \mu') = \frac{1}{2\pi} \int_0^{2\pi} P(\cos\Theta) d\phi = \sum_{l=0}^{N} \tilde{\omega}_l P_l(\mu) P_l(\mu'). \quad (2.2a)$$

The cosine of the scattering angle is defined by $\cos\Theta = \mu\mu' + (1 - \mu^2)^{1/2}(1 - \mu'^2)^{1/2} \cos\phi$, with ϕ the azimuthal angle. The last expression in Eq. (2.2a) may be obtained by using the addition theorem for Legendre polynomials P_l, based on the expansion of the scattering phase function in the form

$$P(\cos\Theta) = \sum_{l=0}^{N} \tilde{\omega}_l P_l(\cos\Theta), \quad (2.2b)$$

where the moment $\tilde{\omega}_l$ can be determined from the orthogonal property of Legendre polynomials,

$$\tilde{\omega}_l = \frac{2l+1}{2} \int_{-1}^{1} P(\cos\Theta) P_l(\cos\Theta) d\cos\Theta. \quad (2.3)$$

In our notations, $\tilde{\omega}_0 = 1$ and $\tilde{\omega}_1/3 = g$, the asymmetry factor.

Replacing the integral by summation, according to the Gauss quadrature, and using the phase function expansion expressed in Eq. (2.2), we obtain

$$\mu_i \frac{dI_i}{d\tau} = I_i - \frac{\tilde{\omega}}{2} \sum_{l=0}^{N} \tilde{\omega}_l P_l(\mu_i) \sum_{j=-n}^{n} I_j P_l(\mu_j) a_j - \frac{\tilde{\omega}}{4\pi} \pi F_0 \times \sum_{l=0}^{N} \tilde{\omega}_l P_l(\mu_i) P_l(-\mu_0) e^{-\tau/\mu_0}, \quad i = -n, n \quad (2.4)$$

where the quadrature point $\mu_{-j} = -\mu_j$, $j \neq 0$, and the weight $a_{-j} = a_j$ and $\sum_{j=-n}^{n} a_j = 2$.

To simplify Eq. (2.4) we define

$$b_{i,j} = \begin{cases} c_{i,j}/\mu_i, & i \neq j \\ (c_{i,j} - 1)/\mu_i, & i = j \end{cases} \quad (2.5)$$

$$c_{i,j} = \frac{\tilde{\omega}}{2} a_j \sum_{l=0}^{N} \tilde{\omega}_l P_l(\mu_i) P_l(\mu_j), \quad i = -n, n, \quad j = -n, -0, n \quad (2.6)$$

where $a_{-0} = 1$ and the notation -0 is used to be consistent with the definition $\mu_{-0} = -\mu_0$. Using the property of Legendre polynomials, we find

$$b_{-i,-j} = -b_{i,j}, \quad b_{-i,j} = -b_{i,-j}. \quad (2.7)$$

Further, we define the direct solar beam in the form

$$I_\odot = e^{-\tau/\mu_0} \pi F_0 / 2\pi. \quad (2.8)$$

Using the preceding definitions, Eq. (2.4) now becomes

$$\frac{dI_i}{d\tau} = -\sum_{j=-n}^{n} b_{i,j} I_j - b_{i,-0} I_\odot, \quad i = -n, n. \quad (2.9)$$

This is the basic discretized equation from which we shall develop the four-stream approximation. The solution for the two-stream approximation can be directly derived from the equation by setting $n = N = 1$.

3. Four-stream approximation for radiative transfer

We consider two radiative streams in the upper and lower hemispheres (i.e., let $n = 2$). At the same time, expand the scattering phase function in four terms, i.e., $N = 3$ in line with the four radiative streams. Using the relation denoted in Eq. (2.7), four first-order differential equations can then be written explicitly in a matrix form:

$$\frac{d}{d\tau} \begin{bmatrix} I_{-2} \\ I_{-1} \\ I_1 \\ I_2 \end{bmatrix} = - \begin{bmatrix} -b_{2,2} & -b_{2,1} & -b_{2,-1} & -b_{2,-2} \\ -b_{1,2} & -b_{1,1} & -b_{1,-1} & -b_{1,-2} \\ b_{1,-2} & b_{1,-1} & b_{1,1} & b_{1,2} \\ b_{2,-2} & b_{2,-1} & b_{2,1} & b_{2,2} \end{bmatrix} \times \begin{bmatrix} I_{-2} \\ I_{-1} \\ I_1 \\ I_2 \end{bmatrix} - \begin{bmatrix} b_{-2,-0} \\ b_{-1,-0} \\ b_{1,-0} \\ b_{2,-0} \end{bmatrix} I_\odot. \quad (3.1)$$

The Gauss quadratures and weights in the four-stream approximation are $\mu_1 = 0.3399810$, $\mu_2 = 0.8611363$, and $a_1 = 0.6521452$, $a_2 = 0.3478548$. The four-by-four matrix represents the contribution of multiple scattering. Thus, the derivative of the diffuse intensity at a specific quadrature angle is the weighted sum of the multiple scattered intensity from all four quadrature angles. The last term represents the contribution of the unscattered component of direct solar flux at position τ.

We shall proceed with a direct approach to find the eigenvalues and eigenvectors for Eq. (3.1). To do so, we define the sum and difference of the upward and downward intensities in the form

$$M_{1,2}^{\pm} = I_{1,2} \pm I_{-1,-2}. \tag{3.2}$$

From Eq. (3.1), we obtain the following four equations:

$$-\frac{dM_2^+}{d\tau} = b_{22}^- M_2^- + b_{21}^- M_1^- + b_2^+ I_\odot, \tag{3.3a}$$

$$-\frac{dM_2^-}{d\tau} = b_{22}^+ M_2^+ + b_{21}^+ M_1^+ + b_2^- I_\odot, \tag{3.3b}$$

$$-\frac{dM_1^+}{d\tau} = b_{12}^- M_2^- + b_{11}^- M_1^- + b_1^+ I_\odot, \tag{3.3c}$$

$$-\frac{dM_1^-}{d\tau} = b_{12}^+ M_2^+ + b_{11}^+ M_1^+ + b_1^- I_\odot, \tag{3.3d}$$

where the coefficients are defined by

$$b_{22}^{\pm} = b_{2,2} \pm b_{2,-2}, \quad b_{21}^{\pm} = b_{2,1} \pm b_{2,-1},$$
$$b_{12}^{\pm} = b_{1,2} \pm b_{1,-2}, \quad b_{11}^{\pm} = b_{1,1} \pm b_{1,-1},$$
$$b_2^{\pm} = b_{2,-0} \pm b_{-2,-0}, \quad b_1^{\pm} = b_{1,-0} \pm b_{-1,-0}. \tag{3.4}$$

Equations (3.3a)–(3.3d) can be combined to yield

$$\frac{d^2}{d\tau^2}\begin{bmatrix} M_2^+ \\ M_1^+ \end{bmatrix} = \begin{bmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{bmatrix}\begin{bmatrix} M_2^+ \\ M_1^+ \end{bmatrix} + \begin{bmatrix} d_2 \\ d_1 \end{bmatrix} I_\odot, \tag{3.5}$$

where

$$a_{22} = b_{22}^+ b_{22}^- + b_{12}^+ b_{21}^-, \quad a_{21} = b_{22}^- b_{21}^+ + b_{21}^- b_{11}^+,$$
$$a_{12} = b_{12}^- b_{22}^+ + b_{11}^- b_{12}^+, \quad a_{11} = b_{12}^- b_{21}^+ + b_{11}^- b_{11}^+,$$
$$d_2 = b_{22}^- b_2^- + b_{21}^- b_1^- + b_2^+/\mu_0,$$
$$d_1 = b_{12}^- b_2^- + b_{11}^- b_1^- + b_1^+/\mu_0. \tag{3.6}$$

Further differential operations in Eq. (3.5) lead to

$$\frac{d^4 M_2^+}{d\tau^4} = b\frac{d^2 M_2^+}{d\tau^2} + cM_2^+ + (d_2/\mu_0^2 + a_{21}d_1 - a_{11}d_2)I_\odot, \tag{3.7a}$$

$$\frac{d^4 M_1^+}{d\tau^4} = b\frac{d^2 M_1^+}{d\tau^2} + cM_1^+ + (d_1/\mu_0^2 + a_{12}d_2 - a_{22}d_1)I_\odot, \tag{3.7b}$$

where $b = a_{22} + a_{11}$ and $c = a_{21}a_{12} - a_{11}a_{22}$. The complete solution for M_2^+ (or M_1^+) is the sum of the solution for the homogeneous part of the fourth-order differential equation plus a particular solution. Thus,

$$\begin{bmatrix} M_2^+ \\ M_1^+ \end{bmatrix} = \sum_{j=-2}^{2}\begin{bmatrix} G_j \\ H_j \end{bmatrix} e^{-k_j\tau} + \begin{bmatrix} \eta_2 \\ \eta_1 \end{bmatrix} e^{-\tau/\mu_0}, \tag{3.8}$$

where G_j and H_j are associated with eigenvectors and η_2 and η_1 are results for the particular solutions. Considering the homogeneous part in Eq. (3.7a) and substituting the homogeneous solution for M_2^+ into this equation, we find

$$\sum_{j=-2}^{2}(k_j^4 - bk_j^2 - c)G_j e^{-k_j\tau} = 0. \tag{3.9}$$

In order to have a nontrivial solution for M_2^+ (or M_1^+), we must have

$$f(k) = k^4 - bk^2 - c = 0. \tag{3.10}$$

It follows that the eigenvalues are given by

$$k^2 = (b \pm \sqrt{b^2 + 4c})/2. \tag{3.11}$$

From the definitions of b and c, we have $b^2 + 4c = (a_{11} - a_{22})^2 + 4a_{21}a_{12}$. The terms a_{21} and a_{12} can be expressed in terms of $c_{i,j}$ defined in Eq. (2.6). Under the conditions that $0 < g < 1$ and $0 < \tilde{\omega} < 1$, we find $a_{21} < 0$ and $a_{12} < 0$. This implies that $b^2 + 4c > 0$. Also, it can be shown that the term

$$c = -\prod_{l=0}^{3}(1 - \tilde{\omega}g^l)/\mu_1^2\mu_2^2,$$

which is less than zero, so that the roots are all real numbers. In the case of conservative scattering, $\tilde{\omega} = 1$, $c = 0$, and the two roots are zero. By substituting the particular solution for $M_{1,2}^+$ into Eqs. (3.7a, b) we obtain

$$\eta_2 = \frac{d_2/\mu_0^2 + a_{21}d_1 - a_{11}d_2}{f(1/\mu_0)}\frac{\pi F_0}{2\pi}, \tag{3.12a}$$

$$\eta_1 = \frac{d_1/\mu_0^2 + a_{12}d_2 - a_{22}d_1}{f(1/\mu_0)}\frac{\pi F_0}{2\pi}. \tag{3.12b}$$

In these equations, the function f is defined in Eq. (3.10). Since G_j and H_j in Eq. (3.8) are defined after high-order differentiations, they are not mutually independent. We may determine their relationship from the homogeneous part of Eq. (3.5). A straightforward substitution yields

$$H_1 e^{-k_1\tau} + H_{-1}e^{k_1\tau} = A_1(G_1 e^{-k_1\tau} + G_{-1}e^{k_1\tau}), \tag{3.13a}$$

$$H_2 e^{-k_2\tau} + H_{-2}e^{k_2\tau} = A_2(G_2 e^{-k_2\tau} + G_{-2}e^{k_2\tau}), \tag{3.13b}$$

where $A_{1,2} = (k_{1,2}^2 - a_{22})/a_{21}$ and k_1 and k_2 are eigenvalues from Eq. (3.11).

Following the preceding procedures and analogs to Eq. (3.7), we may obtain expressions for $M_{1,2}$ in the form

$$\frac{d^4 M_2^-}{d\tau^4} = b'\frac{d^2 M_2^-}{d\tau^2} + c'M_2^- + (d_2'/\mu_0^2 + a_{21}'d_1' - a_{11}'d_2')I_\odot, \tag{3.14a}$$

$$\frac{d^4M_1^-}{d\tau^4} = b'\frac{d^2M_1^-}{d\tau^2} + c'M_1^- + (d'_1/\mu_0^2 + a'_{12}d'_2 - a'_{22}d'_1)I_\odot, \quad (3.14b)$$

where the prime coefficients can be obtained by replacing the superscripts + and − in Eq. (3.6) by − and +, respectively. Also, we note that $b' = a'_{22} + a'_{11} = b$, and $c' = a'_{21}a'_{12} - a'_{11}a'_{22} = c$. The particular solutions for $M_{2,1}^-$ are

$$M_{2,1}^- = \eta'_{2,1}e^{-\tau/\mu_0}, \quad (3.15)$$

with

$$\eta'_2 = \frac{d'_2/\mu_0^2 + a'_{21}d'_1 - a'_{11}d'_2}{f(1/\mu_0)}\frac{\pi F_0}{2\pi}, \quad (3.16a)$$

$$\eta'_1 = \frac{d'_1/\mu_0^2 + a'_{12}d'_2 - a'_{22}d'_1}{f(1/\mu_0)}\frac{\pi F_0}{2\pi}. \quad (3.16b)$$

From Eqs. (3.3b, d), the homogeneous solutions for $M_{2,1}^-$ are given by

$$M_2^- = \frac{A_1b_{21}^- - b_{11}^-}{a^-}k_1(-G_1e^{-k_1\tau} + G_{-1}e^{k_1\tau}) + \frac{A_2b_{21}^- - b_{11}^-}{a^-}k_2(-G_2e^{-k_2\tau} + G_{-2}e^{k_2\tau}), \quad (3.17a)$$

$$M_1^- = \frac{b_{12}^- - A_1b_{22}^-}{a^-}k_1(-G_1e^{-k_1\tau} + G_{-1}e^{k_1\tau}) + \frac{b_{12}^- - A_2b_{22}^-}{a^-}k_2(-G_2e^{-k_2\tau} + G_{-2}e^{k_2\tau}), \quad (3.17b)$$

where $a^- = b_{22}^-b_{11}^- - b_{12}^-b_{21}^-$.

Finally, combining Eqs. (3.8), (3.13) and (3.17), the complete solutions for I_i ($i = -2, -1, 1, 2$) are given by

$$\begin{bmatrix} I_1 \\ I_{-1} \\ I_2 \\ I_{-2} \end{bmatrix} = \begin{bmatrix} \Phi_1^+e^{-k_1\tau} & \Phi_1^-e^{k_1\tau} & \Phi_2^+e^{-k_2\tau} & \Phi_2^-e^{k_2\tau} \\ \Phi_1^-e^{-k_1\tau} & \Phi_1^+e^{k_1\tau} & \Phi_2^-e^{-k_2\tau} & \Phi_2^+e^{k_2\tau} \\ \phi_1^+e^{-k_1\tau} & \phi_1^-e^{k_1\tau} & \phi_2^+e^{-k_2\tau} & \phi_2^-e^{k_2\tau} \\ \phi_1^-e^{-k_1\tau} & \phi_1^+e^{k_1\tau} & \phi_2^-e^{-k_2\tau} & \phi_2^+e^{k_2\tau} \end{bmatrix} \times \begin{bmatrix} G_1 \\ G_{-1} \\ G_2 \\ G_{-2} \end{bmatrix} + \begin{bmatrix} Z_1^+ \\ Z_1^- \\ Z_2^+ \\ Z_2^- \end{bmatrix} e^{-\tau/\mu_0}, \quad (3.18)$$

where the eigenvectors are

$$\phi_{1,2}^{\pm} = \frac{1}{2}\left(1 \pm \frac{b_{11}^- - A_{1,2}b_{21}^-}{a^-}k_{1,2}\right), \quad (3.19a)$$

$$\Phi_{1,2}^{\pm} = \frac{1}{2}\left(A_{1,2} \pm \frac{A_{1,2}b_{22}^- - b_{12}^-}{a^-}k_{1,2}\right). \quad (3.19b)$$

In Eqs. (3.19a–b), $b_{ij}^{\pm}$ are defined by Eq. (3.4), with $b_{i,j}$ given in Eqs. (2.5) and (2.6), and $k_{1,2}$ by roots of Eq. (3.11) with b and c defined below Eq. (3.7b). Expressions below Eqs. (3.17b) and (3.13b) define a^- and $A_{1,2}$, respectively, with a_{ij} given in Eq. (3.6). The Z-functions are defined by

$$Z_{1,2}^{\pm} = \frac{1}{2}(\eta_{1,2} \pm \eta'_{1,2}), \quad (3.19c)$$

where $\eta_{1,2}$ and $\eta'_{1,2}$ are defined by Eqs. (3.12) and (3.16). In Eq. (3.6) d_i and a_{ij} are given and $f(1/\mu_0)$ has the same expression as that in Eq. (3.10), except k is replaced by $1/\mu_0$. In Eq. (3.6) d'_i and a'_{ij} have the same expressions as those except the superscripts + and − are replaced by − and +, respectively. The coefficients G_j ($j = 1, 2, 3, 4$) are to be determined from radiation boundary conditions. Consider a homogeneous layer characterized by an optical depth τ_1 and assume that there is no diffuse radiation from the top and bottom of this layer, then the boundary conditions are

$$\left.\begin{aligned} I_{-1,-2}(\tau = 0) &= 0 \\ I_{1,2}(\tau = \tau_1) &= 0 \end{aligned}\right\}. \quad (3.20)$$

The lower boundary condition can be modified to include the surface albedo effects. Using these boundary conditions, G_j can be obtained by an inversion of a four-by-four matrix in Eq. (3.18). The upward and total (diffuse plus direct) downward fluxes at a given level τ are given by

$$F^+(\tau) = 2\pi(a_1\mu_1I_1 + a_2\mu_2I_2), \quad (3.21a)$$

$$F^-(\tau) = 2\pi(a_1\mu_1I_{-1} + a_2\mu_2I_{-2}) + \mu_0\pi F_0e^{-\tau/\mu_0}. \quad (3.21b)$$

We may also apply the four-stream solutions to inhomogeneous atmospheres in a manner suggested by Liou (1975).

It is possible to incorporate a δ-function adjustment to account for the forward diffraction peak in the context of the four-stream approximation (Cuzzi et al. 1982). In reference to Eq. (2.2), we may express the normalized phase function expansion by incorporating the δ-forward adjustment in the form

$$P_\delta(\cos\Theta) = 2f\delta(\cos\Theta - 1) + (1 - f)\sum_{l=0}^{N}\tilde{\omega}'_lP_l(\cos\Theta), \quad (3.22)$$

where δ denotes the δ-function, f is the fraction of scattered energy residing in the forward peak, and $\tilde{\omega}'_l$ is the adjusted coefficient in the phase function expansion. The forward peak coefficient f in the four-stream approximation can be evaluated by demanding that the next-highest order coefficient in the prime expansion, $\tilde{\omega}'_4$, vanishes (e.g., see Cuzzi et al. 1982). Setting $P(\cos\Theta) = P_\delta(\cos\Theta)$ and utilizing the orthogonal property of Legendre polynomials, we find

$$\tilde{\omega}'_l = [\tilde{\omega}_l - f(2l + 1)]/(1 - f). \quad (3.23)$$

Letting $\tilde{\omega}'_4 = 0$, we obtain $f = \tilde{\omega}_4/9$. Based on Eq. (3.23),

$\tilde{\omega}'_l$ (l = 0, 1, 2, 3) can be evaluated from the expansion coefficients of the phase function $\tilde{\omega}_l$, l = 0, 1, 2, 3, 4.

To incorporate the forward peak contribution in multiple scattering, we consider an adjusted absorption and scattering atmosphere in such a manner that the forward peak contribution f is removed from the optical depth τ, single-scattering parameter $\tilde{\omega}$ and phase function P. The optical (extinction) depth is the sum of the scattering (τ_s) and absorption (τ_a) optical depths. Since the forward peak is only associated with scattering, the adjusted scattering and absorption optical depth must be $\tau'_s = (1-f)\tau_s$ and $\tau'_a = \tau_a$. The total adjusted optical depth is then

$$\tau' = \tau'_s + \tau'_a = \tau(1 - f\tilde{\omega}), \tag{3.24}$$

where the single-scattering albedo $\tilde{\omega} = \tau_s/\tau$. The adjusted single-scattering albedo is defined by

$$\tilde{\omega}' = \frac{\tau'_s}{\tau'} = \frac{(1-f)\tilde{\omega}}{1-f\tilde{\omega}}. \tag{3.25}$$

Moreover, the adjusted phase function from Eq. (3.22) is given by

$$P'(\cos\Theta) = \sum_{l=0}^{N} \tilde{\omega}'_l P_l(\cos\Theta). \tag{3.26}$$

Equations (3.24), (3.25) and (3.26) constitute the generalized similarity principle for radiative transfer. That is, the removal of the forward diffraction peak in scattering processes using adjusted single-scattering parameters is "equivalent" to actual scattering processes. The similarity principle was first stated by Sobolev (1975) for isotropic scattering. The inclusion of the asymmetry factor was discussed by van de Hulst (1980). The principle of employing a Dirac δ-function to approximate highly peaked forward scattering in radiative transfer has been described by Hansen (1969), Potter (1970), and Wiscombe (1977).

4. Computational results and conclusions

Using the analytic equations derived in section 3 via Eqs. (3.21) and (3.18), we compute the reflection r and total transmission t of the solar flux $\mu_0\pi F_0$ defined in the forms

$$r(\mu_0) = F^+(0)/\mu_0\pi F_0, \tag{4.1a}$$

$$t(\mu_0) = F^-(\tau_1)/\mu_0\pi F_0, \tag{4.1b}$$

where all notations have been defined previously. In the computation, the analytic Henyey-Greenstein phase function expanded in the asymmetry factor g,

$$P(\cos\Theta) = \sum_{l=0}^{N} (2l+1) g^l P_l(\cos\Theta), \tag{4.2}$$

was used. The accuracies of δ-two-stream and δ-four-stream approximations are examined by comparing the approximate results with the "exact" values computed from the adding method for radiative transfer. We present illustrative graphs in terms of the relative accuracy in percentage. Let the reflections computed from the approximate and "exact" methods be denoted by $\hat{r}$ and r, respectively. Then the relative accuracy is defined by $\Delta r/r \times 100\% = (\hat{r} - r)/r \times 100\%$. Likewise, the relative accuracy for the total transmission is defined by $\Delta t/t \times 100\%$.

We used numerous asymmetry factors, single-scattering albedos, optical depths and solar zenith angles in the computations. For presentation purposes, however, we select two single-scattering albedos of 1 and 0.8, optical depths from 0.1 to 50 (interval of 0.1 from 0.1 to 1, of 1 from 1 to 10, and 5 from 10 to 50), and cosines of the solar zenith angle from 0 (0.01) to 1 (interval of 0.1). The asymmetry factor chosen for the graphic presentation is 0.75. To highlight the relative accuracy of the presentation, heavy shading is used for accuracies within 5%, while 5%–10% is denoted by light shading. White regions show errors greater than 10%.

Figure 1 shows the relative accuracy of the δ-two-stream (top graphs) and δ-four-stream (bottom graphs) approximations displayed in the intervals of 0%, 1%, 2%, 5%, 10%, etc. The accuracy of the δ-two-stream approximation is comparable to that of the δ-Eddington approximation presented by King and Harshvardhan (1986). For conservative scattering, the reflection values produced by both approximations have low accuracies on the order of 10%–30% for $\mu_0 < 0.5$ and $\mu_0 > 0.9$ with $\tau < 1$. Errors greater than 10% occur for the total transmission when $\mu_0 < 0.2$. In general, reflection and total transmission values computed from the δ-four-stream approximation are within about 5% accuracy, except for three small regions. For reflection, 5%–10% errors occur for $\mu_0 < 0.3$ and $0.6 < \mu_0 < 1$ when $\tau < 1$. For total transmission, errors greater than 5% are produced for very low solar zenith angles ($\mu_0 < 0.1$). It is noted that these regions are associated with very small values. Thus, absolute errors are extremely small (<1%). In the case of $\tilde{\omega} = 0.8$, significant absorption could be built up for large optical depths and/or small solar zenith angles. The δ-two-stream (or δ-Eddington) approximation generally produces errors greater than 5%–10%, as is evident from the graphic presentation. In particular, due to small transmission values, errors of more than 50% can be produced for large optical depths. The δ-four-stream approximation, on the other hand, has a comparable relative accuracy, i.e., within about 5%, to the case of conservative scattering. Errors of 5%–10% occur only for very low solar zenith angles ($\mu_0 < 0.2$).

In addition to the aforementioned results, we have also performed computations using asymmetry factors of 0.7, 0.8 and 0.85 for the analytic Henyey-Greenstein phase function. The actual phase functions for cloud droplets were also employed in accuracy checks, as were the surface albedos. The accuracy of the δ-four-stream

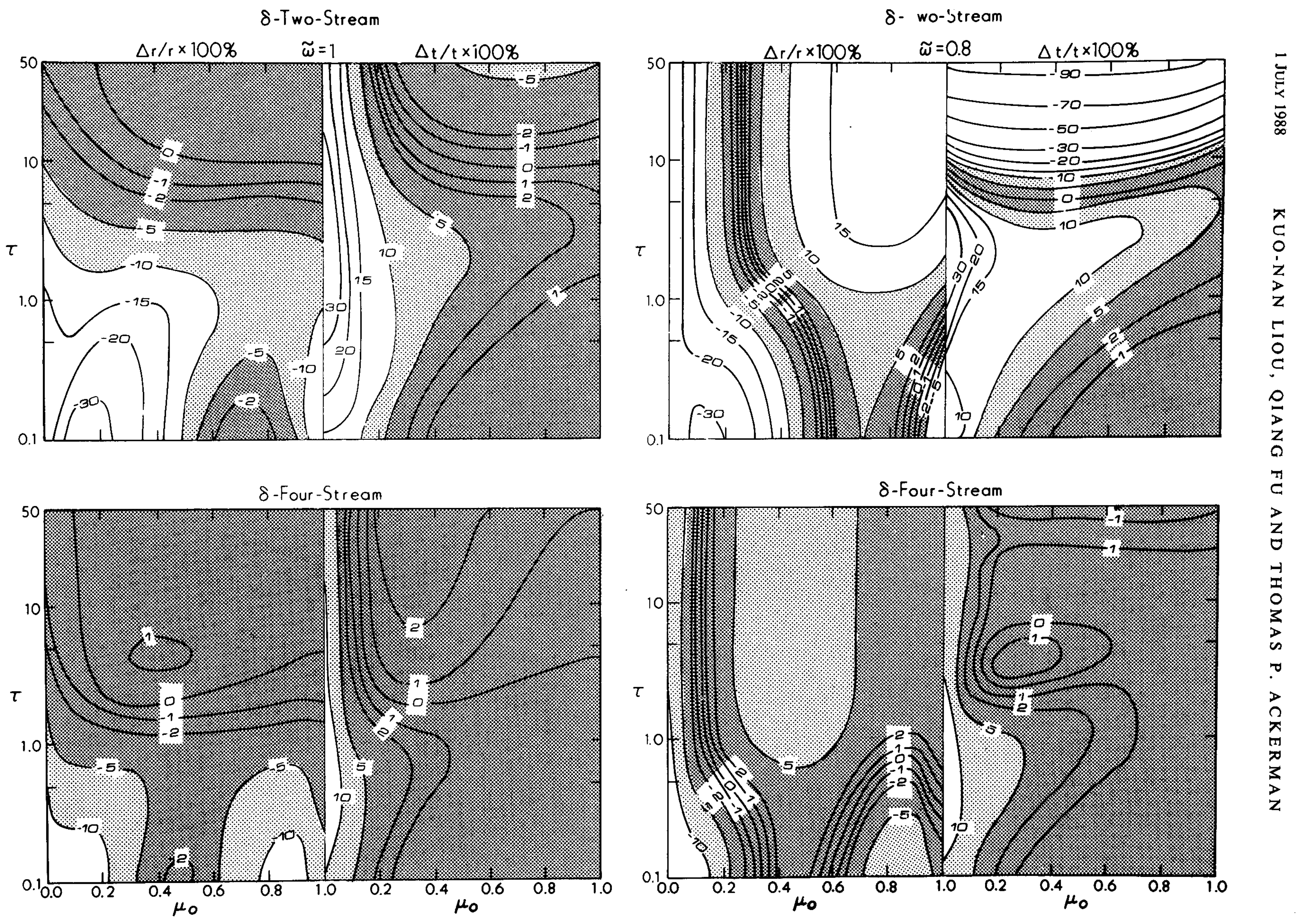

FIG. 1. Relative accuracy of the reflection ($\hat{r}$) and total transmission ($\hat{t}$) computed from the δ-two-stream (upper graphs) and δ-four-stream (lower graphs), approximations with respect to those (r, t) from the adding method for radiative transfer. The relative accuracy is defined by $\Delta r/r = (\hat{r} - r)/r$ for reflection and $\Delta t/t = (\hat{t} - t)/t$ for total transmission. The results are shown in the domain of the optical depth τ and the cosine of the solar zenith angle μ_0, and expressed in terms of percentage. The heavy and light shadings denote errors within 5% and within 5%–10%, respectively, while the white area represents errors greater than 10%. The left and right graphs are, respectively, for $\tilde{\omega} = 1$ (conservative scattering) and $\tilde{\omega} = 0.8$.

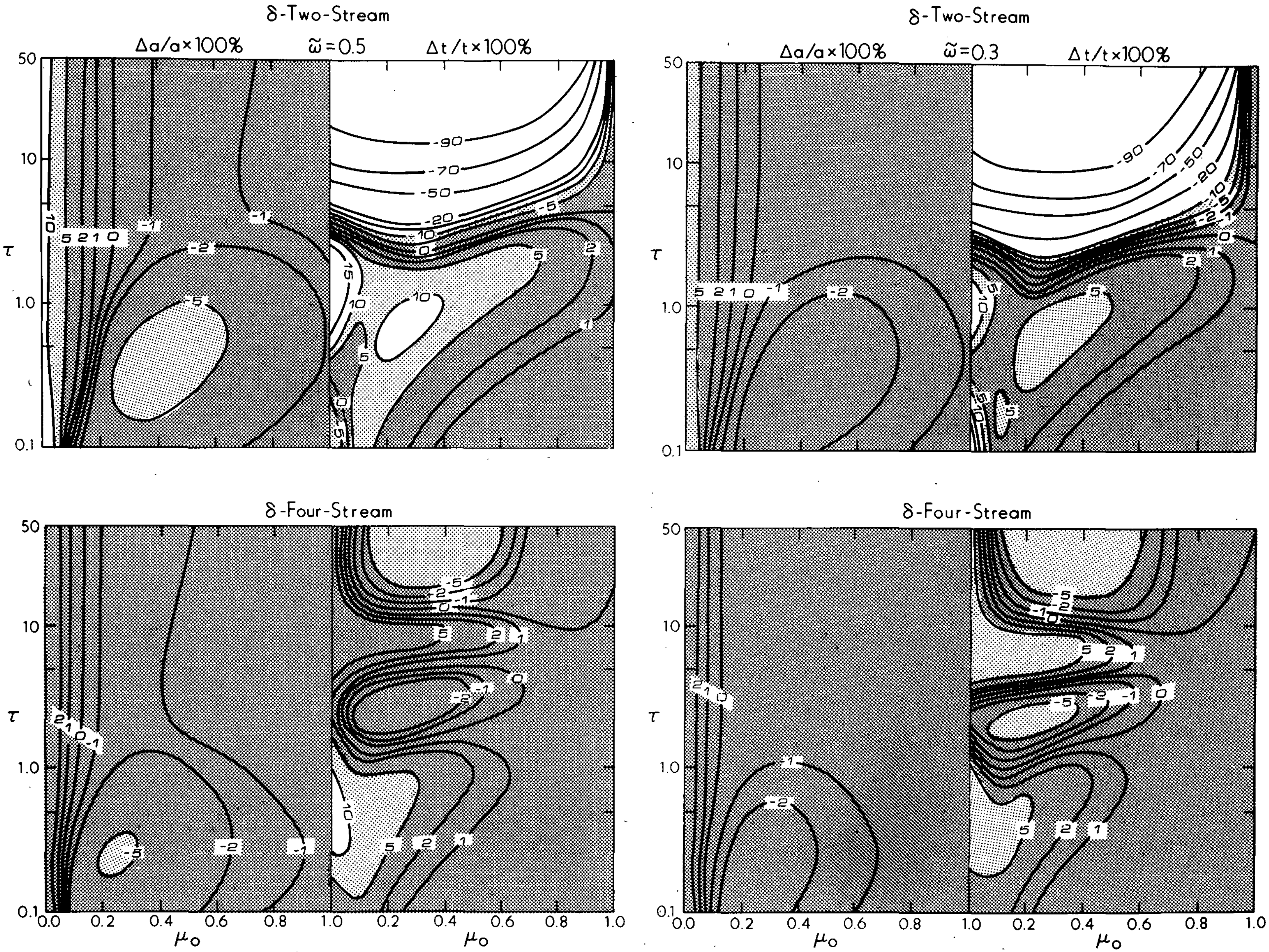

FIG. 2. As in Fig. 1, except for absorption a and total transmission t, where $a = 1 - r - t$. The relative accuracy for absorption is defined by $\Delta\hat{a}/a = (\hat{a} - a)/a$. The left and right graphs are, respectively, for $\tilde{\omega} = 0.5$ and $\tilde{\omega} = 0.3$.

approximation and, for that matter, the δ-two-stream or δ-Eddington approximation is not sensitive to small variations in the asymmetry factor and the detailed structure of the phase function. Also, variations in the surface albedo do not significantly alter the accuracy of the approximations. Finally, we examine the accuracy of δ-two-stream and δ-four-stream approximations in the case of Rayleigh scattering. Since $g = 0$ for Rayleigh atmospheres, there would be no δ-adjustment and use of the two-stream method is equivalent to the isotropic scattering approximation. The four-stream approximation for flux calculations in Rayleigh atmospheres has an accuracy within about 3%.

For applications to solar absorption bands in which gaseous absorption in scattering atmospheres must be accounted for, the single-scattering albedo could be small. We investigate the accuracies of δ-two-stream and δ-four-stream approximations using single-scattering albedos of 0.5 and 0.3. Other parameters remain the same as in Fig. 1. For the cases involving large absorption, the reflection values are generally very small. Thus, we present the percentage of relative accuracy for absorption, $\Delta a/a \times 100\%$ where $a = 1 - r - t$, and total transmission. Figure 2 shows that the δ-two-stream approximation for absorption calculations produces adequate accuracies, which increase as $\tilde{\omega}$ decreases, i.e., absorption increases. The δ-four-stream approximation has a better accuracy than the δ-two-stream with errors for absorption generally less than 2%. It is noted that as $\tilde{\omega}$ decreases, the effects of multiple scattering on flux calculations become less important. For the total transmission, errors from the δ-four-stream approximation are again within about 5%. Large relative errors can be produced by the δ-two-stream approximation when the transmission values are small.

In this paper, we have presented a simple and systematic formulation of the δ-four-stream approximation for solar flux calculations. While all approximate methods for radiative flux transfer have advantages and shortcomings in terms of computational accuracies for different $\tilde{\omega}$, τ and μ_0, we demonstrate that this approximation can achieve relative accuracies within about 5% for all reasonable ranges of the single-scattering parameters at a given wavelength. For computations of solar fluxes covering the entire solar spectrum, the averaged accuracy should also be within about 5%. By virtue of the two intensity streams in the upper hemisphere and two in the lower hemisphere, the δ-four-stream approximation has all the radiative characteristics inherent in the δ-two-stream approximation. The solution of this approximation, like various two-stream methods, is in analytic form so that the computer time involved is minimal. The method can be easily applied to inhomogeneous atmospheres. Obviously, the computer time requirement in this case will depend on the number of homogeneous layers used in the calculation.

For radiative transfer parameterizations in numerical models in which a single radiative transfer approximation is required, the δ-four-stream approximation would be an excellent method. In view of the overall high accuracy and computational economy, it is submitted that the method is well suited for flux and heating rate calculations in all atmospheric conditions.

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