Analytic Two-Stream and Four-Stream Solutions for Radiative Transfer[1]

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1. Introduction

In a previous paper in this journal [Liou (1973), hereafter referred to as DOM], discussions were made on the theoretical and computational aspects of the discrete-ordinate method for radiative transfer with applications to cloudy and hazy atmospheres. We indicated that the discrete-ordinate method is one in which the solutions of radiative transfer in cloud layers may be derived with numerical procedures required only in evaluating the constants of proportionality in the analytic solutions. Consequently, such a method may be employed to obtain simplified radiative transfer approximations whose accuracy can be checked with more exact computations and whose computer time requirements can be limited to a minimum. Therefore, the approximate but reliable solutions may be effectively incorporated into dynamic models as well as climatic studies for the purpose of parameterization for the reflected and transmitted solar fluxes emergent from cloud layers [see, e.g., discussions by Manabe and Strickler (1964), Joseph (1971), Arakawa (1972) and Lacis and Hansen (1974)].

On the basis of the theoretical analyses described in DOM, we found that it is not possible to obtain analytic equations for discrete-streams of more than four without involving complicated numerical methods. This is because of the complexity in solving the simultaneous differential equations and in evaluating the corresponding eigenvalues. The concept of finite discrete-streams for radiative transfer has been noted previously by Chandrasekhar (1950), Lenoble (1956), and recently by Weinman and Guetter (1972) and Zdunkowski and Korb (1974). We also cited other contributors in DOM. The purpose of this note is to derive explicitly the analytic equations in closed forms for cases of the two-stream and four-stream approximations from the exact solutions provided in DOM.

2. Analytic solutions for radiative transfer

The appropriate equation describing the diffuse solar radiation field when the vertical distribution of fluxes is considered may be written as

$$\mu\frac{dI(\tau,\mu)}{d\tau}=I(\tau,\mu)-\frac{1}{2}\int_{-1}^{+1}p(\mu,\mu')I(\tau,\mu')d\mu'-\tfrac{1}{4}F_0p(\mu,\mu_0)\exp(-\tau/\mu_0), \tag{1}$$

where I denotes intensity, τ the optical depth, F_0 the solar flux, and μ and μ_0 the cosine of the emergent and the solar zenith angles respectively. Expanding the phase function p in Legrendre polynomials P_l and noting that the cosine of the scattering angle $(\cos\Theta)$ can be denoted as $\mu\mu'$ in the azimuth-independent case, we have

$$p(\cos\Theta)=\sum_{l=0}^{N}\tilde{\omega}_lP_l(\cos\Theta)=\sum_{l=0}^{N}\tilde{\omega}_lP_l(\mu)P_l(\mu'), \tag{2}$$

where the coefficients $\tilde{\omega}_l$ can be determined by noting the orthogonal property of Legendre polynomials:

$$\tilde{\omega}_l=\frac{2l+1}{2}\int_{-1}^{+1}p(X)P_l(X)dX. \tag{3}$$

In the case of the two-stream approximation, $N=1$ in Eq. (2). For the four-stream approximation, however, it is necessary to expand the phase function in four-term polynomials (i.e., $N=3$) to obtain higher accuracy.

The boundary conditions for the diffuse solar radiation in the atmosphere may be taken as

$$\left.\begin{aligned}I(0,-\mu)&=0\\ I(\tau_N,\mu)&=0\end{aligned}\right\}, \tag{4}$$

i.e., no diffuse radiation from the top and the bottom of the cloud layer whose optical depth is τ_N. Eqs. (1), (2) and (4) form the basis in deriving the following analytical solutions.

a. Two-stream approximation

Replacing the integral term in Eq. (1) by summation according to the Gaussian quadrature of two, the up-

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ward and downward flux at any given level can be readily written as

$$F\uparrow(\tau)=2\pi\mu_1[L_1W_1(\mu_1)\exp(-k_1\tau)+L_{-1}W_1(-\mu_1)\times\exp(k_1\tau)+Z(\mu_1)\exp(-\tau/\mu_0)], \quad (5)$$

$$F\downarrow(\tau)=-2\pi\mu_1[L_1W_1(-\mu_1)\exp(-k_1\tau)+L_{-1}W_1(\mu_1)\times\exp(k_1\tau)+Z(-\mu_1)\exp(-\tau/\mu_0)]-\pi\mu_0F_0\exp(-\tau/\mu_0), \quad (6)$$

where:

$$\mu_1=(\tfrac{1}{3})^{\frac{1}{2}}$$

$$L_1=\frac{1}{2}\left[\frac{C^+}{A^+}+\frac{C^-}{A^-}\right] \quad (7)$$

$$L_{-1}=\frac{1}{2}\left[\frac{C^+}{A^+}-\frac{C^-}{A^-}\right]\exp(-k_1\tau_N) \quad (8)$$

$$A^\pm=W_1(-\mu_1)\pm W_1(\mu_1)\exp(-k_1\tau_N) \quad (9)$$

$$C^\pm=-[Z(-\mu_1)\pm Z(\mu_1)\exp(-\tau_N/\mu_0)] \quad (10)$$

$$k_1=\frac{1}{\mu_1}[(1-\tilde{\omega}_0)(1-\tilde{\omega}_1\mu_1^2)]^{\frac{1}{2}} \quad (11)$$

$$W_1(\mu_1)=\frac{1}{1+\mu_1k_1}[\tilde{\omega}_0-\tilde{\omega}_1(1-\tilde{\omega}_0)\mu_1/k_1] \quad (12)$$

$$Z(\mu_1)=\frac{\mu_0F_0}{4}\frac{(\mu_1-\mu_0)[\tilde{\omega}_0-\tilde{\omega}_1(1-\tilde{\omega}_0)\mu_1\mu_0]}{\mu_1^2(1-k_1^2\mu_0^2)}. \quad (13)$$

b. Four-stream approximation

Following the theoretical analyses described in DOM, it is also possible to derive explicitly the analytic solution for the transfer of solar radiation with discrete-streams of four when higher accuracy is required. With the phase function of cloud particles characterized by the four-term expansion in Legendre polynomials, the upward and downward flux at any level τ in clouds are given by

$$F\uparrow(\tau)=2\pi[a_1\mu_1I(\tau,\mu_1)+a_2\mu_2I(\tau,\mu_2)], \quad (14)$$

$$F\downarrow(\tau)=-2\pi[a_1\mu_1I(\tau,-\mu_1)+a_2\mu_2I(\tau,-\mu_2)]-\pi\mu_0F_0\exp(-\tau/\mu_0). \quad (15)$$

In Eqs. (14) and (15), the emergent intensity

$$I(\tau,X)=\sum_{j=1}^{2}[L_jW_j(X)\exp(-k_j\tau)+L_{-j}W_j(-X)\exp(k_j\tau)]+Z(X)\exp(-\tau/\mu_0), \quad (16)$$

where the argument X stands for either value of $\pm\mu_{1,2}$.

The cosine of the discrete-emergent angles $\mu_1=0.3399810$ and $\mu_2=0.8611363$, the weighing factors $a_1=0.6521452$ and $a_2=0.3478548$, and

$$L_{1,2}=(M_{1,2}+N_{1,2})/2 \quad (17)$$

$$L_{-1,-2}=(M_{1,2}-N_{1,2})\exp(-k_{1,2}\tau_N)/2 \quad (18)$$

$$M_1=(C_1^+B_2^+-C_2^+B_1^+)/(A_1^+B_2^+-A_2^+B_1^+) \quad (19)$$

$$M_2=-(C_1^+A_2^+-C_2^+A_1^+)/(A_1^+B_2^+-A_2^+B_1^+) \quad (20)$$

$$N_1=(C_1^-B_2^--C_2^-B_1^-)/(A_1^-B_2^--A_2^-B_1^-) \quad (21)$$

$$N_2=-(C_1^-A_2^--C_2^-A_1^-)/(A_1^-B_2^--A_2^-B_1^-) \quad (22)$$

$$A_{1,2}^\pm=W_1(-\mu_{1,2})\pm W_1(\mu_{1,2})\exp(-k_1\tau_N) \quad (23)$$

$$B_{1,2}^\pm=W_2(-\mu_{1,2})\pm W_2(\mu_{1,2})\exp(-k_2\tau_N) \quad (24)$$

$$C_{1,2}^\pm=-[Z(-\mu_{1,2})\pm Z(\mu_{1,2})\exp(-\tau_N/\mu_0)] \quad (25)$$

$$W_{1,2}(X)=\frac{1}{1+k_{1,2}X}\sum_{l=0}^{3}\tilde{\omega}_l\xi_l(k_{1,2})P_l(X) \quad (26)$$

$$Z(X)=\frac{\mu_0F_0}{4(\mu_0+X)}\frac{(\mu_1^2-\mu_0^2)(\mu_2^2-\mu_0^2)}{\mu_1^2\mu_2^2(1-k_1^2\mu_0^2)(1-k_2^2\mu_0^2)}\times\sum_{l=0}^{3}\tilde{\omega}_l\xi_l\left(\frac{1}{\mu_0}\right)P_l(X). \quad (27)$$

The polynomials and ξ functions for an argument X are respectively as follows:

$$\left.\begin{aligned}P_0(X)&=1\\P_1(X)&=X\\P_2(X)&=\tfrac{1}{2}(3X^2-1)\\P_3(X)&=\tfrac{1}{2}(5X^3-3X)\end{aligned}\right\}, \quad (28)$$

$$\left.\begin{aligned}\xi_0(X)&=1\\\xi_1(X)&=-(1-\tilde{\omega}_0)/X\\\xi_2(X)&=(3-\tilde{\omega}_1)(1-\tilde{\omega}_0)/(2X^2)-\tfrac{1}{2}\\\xi_3(X)&=-(5-\tilde{\omega}_2)(3-\tilde{\omega}_1)(1-\tilde{\omega}_0)/(6X^3)\\&\quad+[(5-\tilde{\omega}_2)+4(1-\tilde{\omega}_0)]/(6X)\end{aligned}\right\}. \quad (29)$$

By noting that (see Chandrasekhar, 1950, pp. 62)

$$\left.\sum_j a_j\mu_j^l=\frac{2\delta_l}{l+1},\quad \begin{matrix}\delta_l=1, & l=\text{even}\\ \delta_l=0, & l=\text{odd}\end{matrix}\right\}, \quad (30)$$

the eigenvalues k's can be determined from

$$k^2=-\frac{b}{2}\pm\tfrac{1}{2}(b^2-4c)^{\frac{1}{2}}, \quad (31)$$

where

$$b=(a_1t_1-1)/\mu_1^2+(a_2t_2-1)/\mu_2^2, \quad (32)$$

$$c=(1-a_1t_1-a_2t_2)/(\mu_1^2\mu_2^2)+(a_1t_1'/\mu_1^2)+(a_2t_2'/\mu_2^2), \quad (33)$$

$$t_{1,2}=\tilde{\omega}_0+\tilde{\omega}_1(1-\tilde{\omega}_0)\mu_{1,2}P_1(\mu_{1,2})-\tfrac{1}{2}\tilde{\omega}_2P_2(\mu_{1,2})-\tfrac{1}{6}\tilde{\omega}_3[(5-\tilde{\omega}_2)+4(1-\tilde{\omega}_0)]\mu_{1,2}P_3(\mu_{1,2}), \quad (34)$$

$$t_{1,2}'=\tfrac{1}{2}(3-\tilde{\omega}_1)(1-\tilde{\omega}_0)[\tilde{\omega}_2P_2(\mu_{1,2})+\tfrac{1}{3}\tilde{\omega}_3(5-\tilde{\omega}_2)\mu_{1,2}P_3(\mu_{1,2})]. \quad (35)$$

It should be noted that when $\tilde{\omega}_0\equiv 1$ (conservative scattering), one of the eigenvalues becomes zero. As a result of singularity, neither Eq. (26) nor (12) can be evaluated. However, by assuming $\tilde{\omega}_0=0.99999$, the above formulas are all valid and errors produced are practically insignificant.

3. Some remarks

The equations presented in Section 2 which involves only simple algebra were programmed for values of the reflection $r[=F\uparrow(0)/\pi\mu_0F_0]$ and the transmission $t[=F\downarrow(\tau_N)/\pi\mu_0F_0]$ in the computer to check with the numerical results discussed in DOM. The computer time required to obtain the reflection and transmission for a number of optical thicknesses τ_N and solar zenith angles μ_0 is on the order of a few seconds with the CDC 6400 computer. The accuracies of the two-stream and four-stream approximations as discussed in DOM are within 3–10% and 1% respectively.

In cases of Rayleigh scattering, $\omega_1=\omega_3=0$, the previous equations in Section 2 for both approximations are reduced to somewhat simpler forms. However, since the two-stream approximation produces low accuracy particularly for optically thin layers, it would be advisable to employ the four-stream approximation which also correctly describes the phase function of Rayleigh scattering.

From the previous formulas, the upward and downward flux for any given level τ in clouds depend on the solar flux πF_0, the solar zenith angle μ_0, the coefficients $\tilde{\omega}_l$ (evaluated from the phase function and the single scattering albedo), and the optical depth τ_N of a cloud layer. The last two variables are to be determined from the composition and structure of clouds, and are functions of the wavelengths in the solar spectrum. At a wavelength of 0.55 μm, values of $\tilde{\omega}_1$, $\tilde{\omega}_2$ and $\tilde{\omega}_3$ are about 2.55, 3.85 and 4.50 respectively for fair weather cumulus based on Mie computations. For clouds composed of larger particles such as cumulonimbus, values of $\tilde{\omega}_l$ increase slightly.

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