Oceanic Dynamical and Material Equations and General Circulation Models

The ocean is a rotating, stratified, weakly compressible fluid with a functionally complex equation of state (EOS) for seawater.

The primary variables are velocity $\mathbf{u}$, pressure p, density ρ, temperature T, salinity S, and various material concentrations C (a.k.a. tracers).

The oceanic domain is bounded from above by a free-surface interface between air and water and from below by a solid bottom surface of silt, sand, and rock with some localized inflows by rivers and hydrothermal vents. With minor departures the interface is immiscible; the bottom is unmoving; and there are no side boundaries.

The forcing for oceanic currents is mainly through the surface interface, usually expressed as horizontal wind stress vector τ^s, heat flux Q, and freshwater flux $\mathcal{F}$ (including rivers with shallow inflows). There is a bottom stress τ^b that arises as an internal product of the circulation (mainly a drag force), but generally negligible bottom heat and water fluxes. For various C materials, there are both surface and bottom fluxes.

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Concept: The fundamental equations of oceanic circulation are known, and they can be solved approximately in computational simulations that are often very realistic.

This is in contrast to most other natural sciences. Even the atmosphere has poorly determined cloud physics that make its models less fundamentally grounded.

The least certain physical component of an oceanic model is its interaction with sea ice, which occurs in only a small fractional volume.

On the other hand, the uncertainties are large in the representation of oceanic biogeochemical processes, especially the organic ones.

The biggest deficiency in oceanic models is their incomplete spatial resolution due to computational limits.

A Fluid-Dynamical Hierarchy

Parameters that measure the dynamical influences of planetary rotation and stable density stratification are the Rossby and Froude numbers,

$$Ro = \frac{V}{fL} \quad \text{and} \quad Fr = \frac{V}{NH}.$$

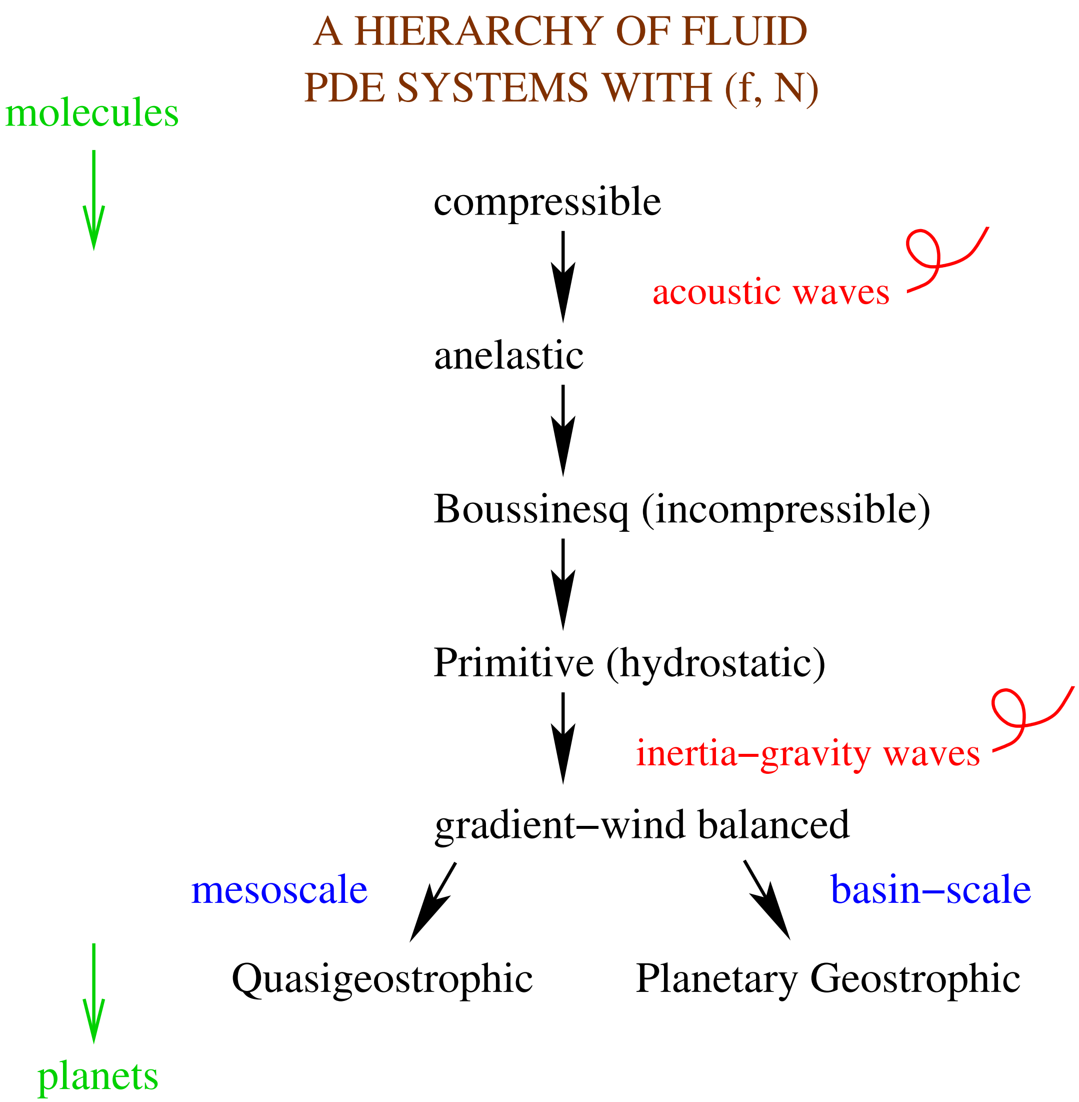
V is a characteristic horizontal velocity; $f = 2\Omega \sin[\varphi]$ is Coriolis frequency (Ω is Earth's rotation rate; φ is latitude); $N \approx [-(g/\rho)\, \partial_z \rho\,]^{1/2}$ (with g the gravitational acceleration; ρ is potential density to eliminate effects of compressibility) is buoyancy frequency for the stratification; and $(L,\ H)$ are (horizontal, vertical) length scales. For flows on the planetary scale and mesoscale, Ro and Fr are $\lesssim 1$. For these flows the atmosphere and ocean are relatively thin, so their aspect ratio,

$$\Lambda = \frac{H}{L},$$

is often small. A measure of the dynamical influence (vs. EOS influence) of compressibility is the Mach number,

$$M = \frac{V}{c_s},$$

where c_s is the speed of sound; $c_s \approx 1500$ m s^{-1}, so M is extremely small.



A dynamical hierarchy of fluid-dynamical equations, approximately ordered by decreasing values of M, Λ, Ro, and Fr and increasing characteristic space and time scales. The branch at the end is between Quasigeostrophic Equations — suitable for coastal and topographic waves, mesoscale eddies, and narrow boundary currents – and Planetary Geostrophic Equations — suitable for large-scale Rossby waves, large wind gyres (outside the western boundary current region), and thermohaline circulation.

Except for sound waves, Boussinesq is sufficient for the ocean. Most general circulation models (OGCMs) use Primitive; many theories (analytical solutions) are geostrophic and exclude inertial and gravity waves, or else are linearized wave equations.

Basic Oceanic Dynamics: the Boussinesq Equations

$$
\begin{aligned}
\frac{D\mathbf{u}}{Dt} + 2\mathbf{\Omega} \times \mathbf{u} &= -\nabla\phi - \frac{g}{\rho_0}(\rho - \rho_0)\hat{\mathbf{z}} + \mathcal{D}[\mathbf{u}] \quad \textit{momentum} \\
\nabla \cdot \mathbf{u} &= 0 \qquad \textit{incompressible mass} = \textit{volume} \\
\frac{DT}{Dt} &= \mathcal{D}[T] \qquad \textit{heat} \\
\frac{DS}{Dt} &= \mathcal{D}[S] \qquad \textit{salinity} \\
\rho &= \mathcal{E}[T, S, z] \qquad \textit{equation of state} \\
\frac{DC}{Dt} &= \mathcal{S}_C + \mathcal{D}[C] \qquad \textit{material concentration}\,,
\end{aligned}
$$

with the advective time derivative for any quantity a,

$$
\frac{Da}{Dt} = \frac{\partial a}{\partial t} + \mathbf{u} \cdot \nabla a\,.
$$

$\phi = p/\rho_o$ is the geopotential function with p the dynamic pressure. $\hat{\mathbf{z}}$ is an upward-vertical unit vector (opposite to gravity). ($\hat{\mathbf{x}}$,$\hat{\mathbf{y}}$) are (east,north).

$\mathcal{E}$ is the EOS, where we have made the fairly accurate approximation that the compressibility is primarily a bulk effect due to the mean hydrostatic pressure at depth $-z$, *viz.*, $p \approx p_{atm} - g\rho_0 z$. Because $\rho_0 \approx 10^3$ kg m^{-3} for seawater, the added pressure at $d = 10$ m is equivalent to the average surface air pressure, $p_{atm} \approx 10^5$ Pa. Otherwise, $\mathcal{E}$ is a complicated functional that is fit to experimental data without a clear theoretical justification.

$\mathcal{D}$ is the "mixing" operator due to averaging over molecular kinetics and unresolved (subgrid-scale) fluid mixing often called eddy diffusion. Its fundamental definition is

$$\mathcal{D}[a] = \kappa_m \nabla^2 a - \nabla \cdot \overline{\mathbf{u}' a'} \,,$$

if we view the governing equations as based on averaging over a certain minimum space and time scale. $\kappa_m > 0$ is a molecular diffusion coefficient for a. In model solutions, the subgrid eddy fluxes are unknown and must be parameterized, often as

$$\mathcal{D}[a] \;\approx\; \left[\kappa_h (\partial_x^2 + \partial_y^2) + \kappa_v \partial_z^2 \right] \, a \,.$$

h and v denote horizontal and vertical, respectively. Typically, $\kappa_m \ll \kappa_v \ll \kappa_h$ for eddy mixing due to the pervasiveness of small Λ effects in the ocean, while $\kappa_v = \kappa_h$ for molecular diffusion. The molecular κ values may be different for different quantities (*e.g.*, $\kappa_{Sm} \ll \kappa_{Tm}$ for molecular diffusion, giving rise to salt fingers and double diffusion at small scale), but the same eddy κ values are used for all materials insofar as they are commonly mixed by the turbulent currents.

Only the C equation has an important interior source/sink term $\mathcal{S}_C$. It represents non-conservative biogeochemical reactions or particulate physics. We might also include such a term for heat due to solar radiation absorbed below the surface, but its important penetration depth is only $\mathcal{O}$(m), so it is often lumped with Q at the surface. Light useful for photosynthesis penetrates much deeper.

Eddy Fluxes

When an average and fluctuations are defined ($e.g.,$ time average during a statistically stationary equilibrium state), then for any quantity a,

$$a = \overline{a} + a' \qquad \text{and} \qquad \overline{a'} = 0.$$

Thus, for any quadratic quantity, in particular an advective flux,

$$\overline{\mathbf{u}a} = \overline{(\overline{\mathbf{u}} + \mathbf{u}')\,(\overline{a} + a')} = \overline{\mathbf{u}}\,\overline{a} + \overline{\mathbf{u}'a'}\,.$$

For example, if $a = \sin[t]$, $\overline{a} = 0$, and $\overline{a^2} = \overline{a'^2} = 1/2$.

In balances with advective time derivative, we can rewrite advection as a flux divergence,

$$(\mathbf{u} \cdot \nabla)a = \nabla \cdot (\mathbf{u}a) - a\nabla \cdot \mathbf{u} = \nabla \cdot (\mathbf{u}a)\,.$$

Hence, an average of advection includes an eddy flux divergence:

$$\overline{(\mathbf{u} \cdot \nabla)a} = \overline{\mathbf{u}} \cdot \nabla\overline{a} + \nabla \cdot \overline{\mathbf{u}'a'}\,.$$

Typically in $\mathcal{D}$ the eddy fluxes are much larger than the molecular diffusion by a ratio we can scale estimate as

$$\frac{\nabla \cdot \overline{\mathbf{u}'a'}}{\kappa\nabla^2\overline{a}} \sim \frac{VA/L}{\kappa A/L^2} = \frac{VL}{\kappa} \gg 1\,,$$

analogous to a Reynolds number for momentum diffusion ($Re = VL/\nu$, with ν the viscosity).

Boundary Conditions

Boundary conditions have zero normal velocity and specified normal fluxes. The kinematic condition at the free surface is

$$w = \frac{D\eta}{Dt} \quad \text{at} \quad z = \eta(x,y,t)\,,$$

which plays an essential role, *e.g.*, in surface gravity waves. w is vertical velocity, and η is the surface elevation, where there is pressure continuity, $p_{ocean} \approx p_{atm}$. At the solid bottom the kinematic condition is

$$w = -\,\mathbf{u}\cdot\nabla h \quad \text{at} \quad z = -\,h(x,y)\,,$$

where h is the bottom depth relative to a resting sea surface at $z = 0$.

The surface flux conditions have the following forms:

$$\begin{aligned} \rho_0 \kappa_u \frac{\partial \mathbf{u}_h}{\partial z} &= \tau^s\,, \qquad \rho_0 C_p \kappa_{Tv} \frac{\partial T}{\partial z} = Q\,, \\ \frac{1}{S_0} \kappa_{Sv} \frac{\partial S}{\partial z} &= -\,\mathcal{F}\,, \qquad \kappa_{Cv} \frac{\partial C}{\partial z} = \mathcal{C}\,, \end{aligned}$$

where τ^s is surface stress [N m^{-2}]; Q is heat flux [W m^{-2}]; $C_p \approx 4000$ J K^{-1} kg^{-1} is heat capacity; $S_0 \approx 35$ PSU is an average S value; $\mathcal{F}$ [m s^{-1}] is vertical freshwater flux per unit horizontal area; and $\mathcal{C}$ [m C s^{-1}] is the surface C flux. All of these fluxes have the sign convention of being positive when they are putting their relevant stuff (momentum, heat, *etc.*) into the ocean. The surface flux fields must be specified as external information.

There are analogous bottom flux relations, in particular a bottom stress τ^b. At both vertical boundaries, the sense of τ is as a drag of the fluid or boundary above on the fluid or boundary below.

Common Dynamical Approximations and Diagnostic Relations

EOS: A local Taylor series expansion of the EOS around a given state (T_0, S_0, d_0) is

$$\rho \;\approx\; \rho_0 \left(1 - \tilde{\alpha}(T - T_0) + \tilde{\beta}(S - S_0) - \tilde{\gamma}(z - z_0) \right) + \ldots \,, \qquad \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \;>\; 0 \,,$$

which is valid for small ΔS, ΔT, and Δz. $\rho_0 \approx 1.25 \times 10^3$ kg m^{-3} is a mean density of seawater; T_0, S_0, and z_0 are local reference values; and $\tilde{\alpha} \leq 2 \times 10^{-4}$ K^{-1} and $\tilde{\beta} \approx 8 \times 10^{-4}$ PSU^{-1} are the thermal-expansion and haline-compression coefficients; the bulk compressibility coefficient is $\tilde{\gamma} \approx 4 \times 10^{-6}$ m^{-1}. These expansion/compression coefficients vary with conditions and location, most substantially for α, which becomes very small at cold polar and abyssal T values. This EOS simplification is not made in OGCMs.

The difference between *in situ* temperature and potential temperature T is often ignored unless atypically large vertical parcel displacements of $\mathcal{O}(km)$ are involved. Ditto for *in situ* density ρ and potential density.

In **hydrostatic** models with a vertical momentum balance,

$$\partial_z \phi = b = -\, g(\rho/\rho_0 - 1) \,,$$

and the EOS above, only horizontal pressure gradients are dynamically important, hence $\tilde{\gamma}$ can be ignored. b is “buoyancy”.

Coriolis force: The Coriolis force is often approximated by neglecting its local horizontal component:

$$2\mathbf{\Omega} \times \mathbf{u} \;\approx\; f(\varphi)\,\hat{\mathbf{z}} \times \mathbf{u}_h\,,$$

where $\mathbf{u}_h$ is the horizontal velocity and φ is the latitude; this is justified by $\Lambda \ll 1$ and is even done in OGCMs. Common further approximations of f are

$$f \;\approx\; f_0$$

(f-plane), with $f_0 = 2|\mathbf{\Omega}|\sin[\varphi_0]$, and

$$f \;\approx\; f_0 + \beta(y - y_0)$$

(β-plane) with $\beta = (2|\mathbf{\Omega}|/R)\cos[\varphi_0]$. $R \approx 6.35 \times 10^6$ m is the radius of Earth, and $y = R\varphi$ is a local Cartesian meridional coordinate (with $x = R\cos[\varphi_0]\lambda$ as its zonal counterpart, where λ is longitude). Note that f and f_0 change sign across the equator, whereas β is always positive and largest at the equator. When $\varphi = f_0 = 0$, the approximation is called the equatorial β-plane. Rossby waves are a common phenomenon in the ocean, and they occur only in models with $\beta \neq 0$.

Geostrophic balance: In the horizontal momentum equations when $Ro \ll 1$ and D/Dt and $\mathcal{D}$ are $\mathcal{O}(Ro)$ relative to the Coriolis and pressure gradient forces,

$$f v_g \approx \ \partial_x \phi \,, \qquad f u_g \approx \ -\partial_y \phi \,;$$

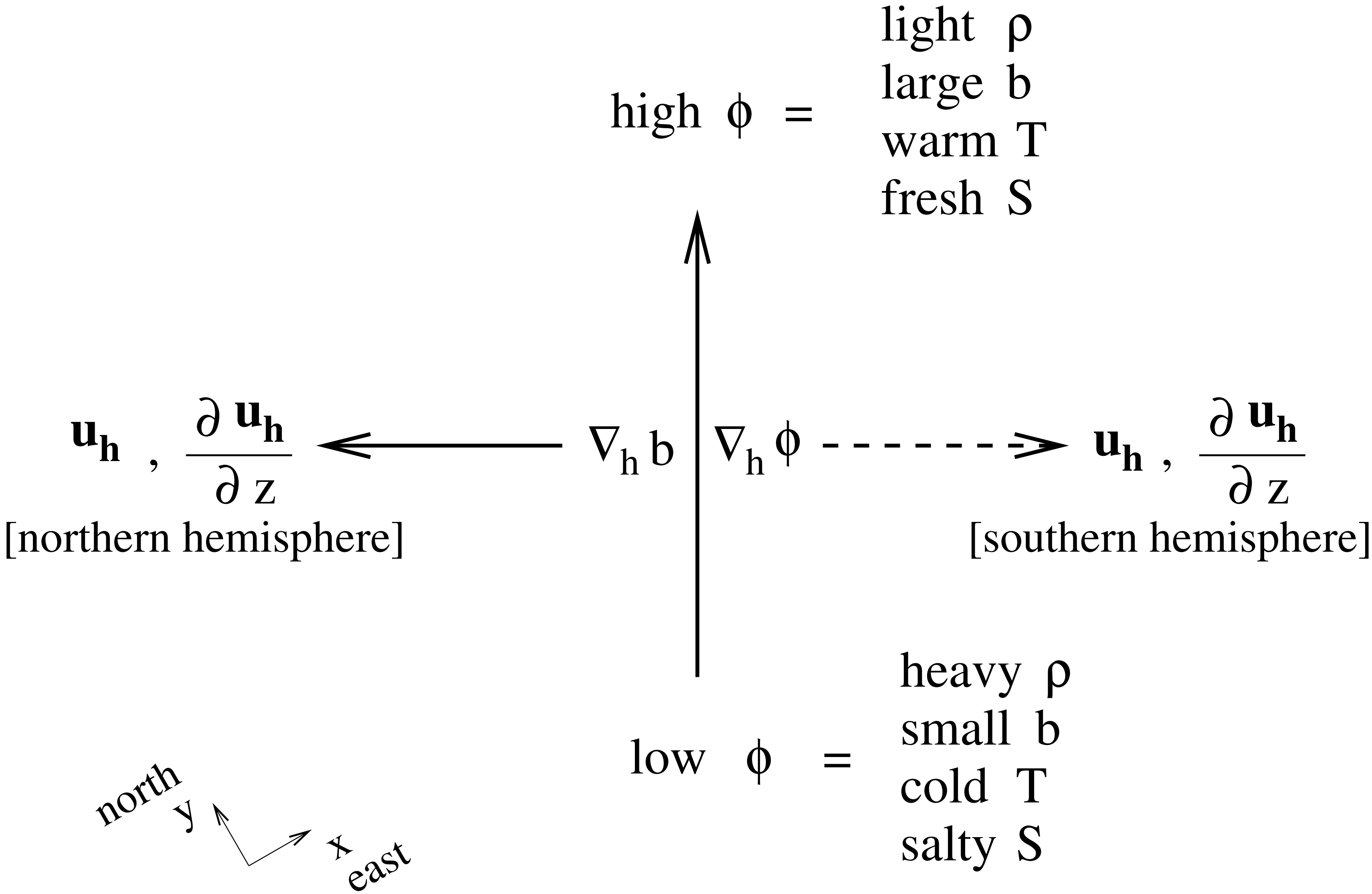
$i.e.$, the geostrophic flow $\mathbf{u}_g$ is along isolines of ϕ in horizontal planes. In combination with hydrostatic balance, this gives the "thermal wind":

$$f \partial_z v_g \approx \ \partial_x b \,, \qquad f \partial_z u_g \approx \ -\partial_y b \,;$$

In combination with an f-plane approximation, it implies non-divergent flow,

$$\partial_x u_g + \partial_y v_g \ \approx \ 0 \,,$$

and $w \approx 0$ to this level of approximation. Because b is known when T and S are measured (and even T is often enough), thermal wind enables an approximate inference of current.



Sketch of the directions of horizontal flow $\mathbf{u}_h$ and its vertical shear $\partial_z \mathbf{u}_h$ (these are equivalent in the common situation of a surface-intensified current) in relation to horizontal gradients in ρ, b, ϕ, T, and S.

Primitive Equations with "Traditional" Coriolis Approximation: These are based on a hydrostatic vertical momentum balance and $\mathbf{\Omega} = f(\varphi)\hat{\mathbf{z}}$. They comprise the fluid dynamical component of most OGCMs.

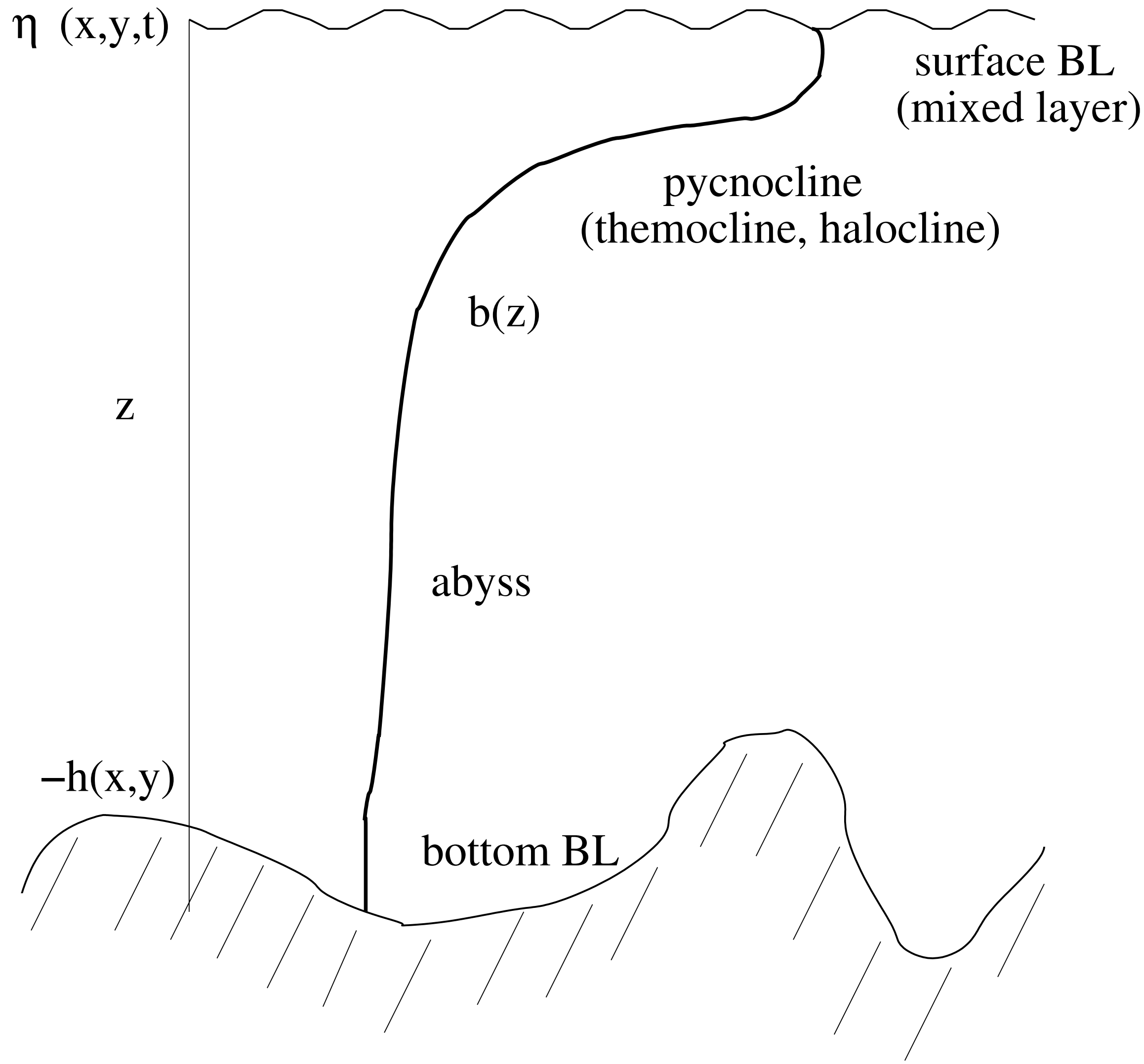
Side Boundaries: The ocean has no large side boundaries as a matter of geophysical reality, but oceanic models often do. A typical continental margin has the shape of a shallow shelf region (with depth $H = \mathcal{O}(100)$ m) extending from the shoreline to the shelf-break (with width $L = \mathcal{O}(10)$ km). Further offshore the continental slope falls sharply to mid-ocean depths ($H = \mathcal{O}(5000)$ m). Thus, unless we are prepared to resolve the shoaling slope and shelf topography slope, a vertical side wall must be assumed. For "climate" OGCMS with typical horizontal grid resolution $\Delta x \approx 100$ km, this is the common practice. Obviously, it is not adequate for coastal currents.

Rigid Lid: Similarly, the ocean has a deformable free surface controlled by the gravitational force, with its kinematic boundary condition and an accompanying pressure continuity condition, $\phi = p_{atm}/\rho_0$ at $z = \eta$. However, on periods long compared to surface gravity wave oscillations ($e.g., > \mathcal{O}(\sim 10)$ s), these may often be accurately approximated by

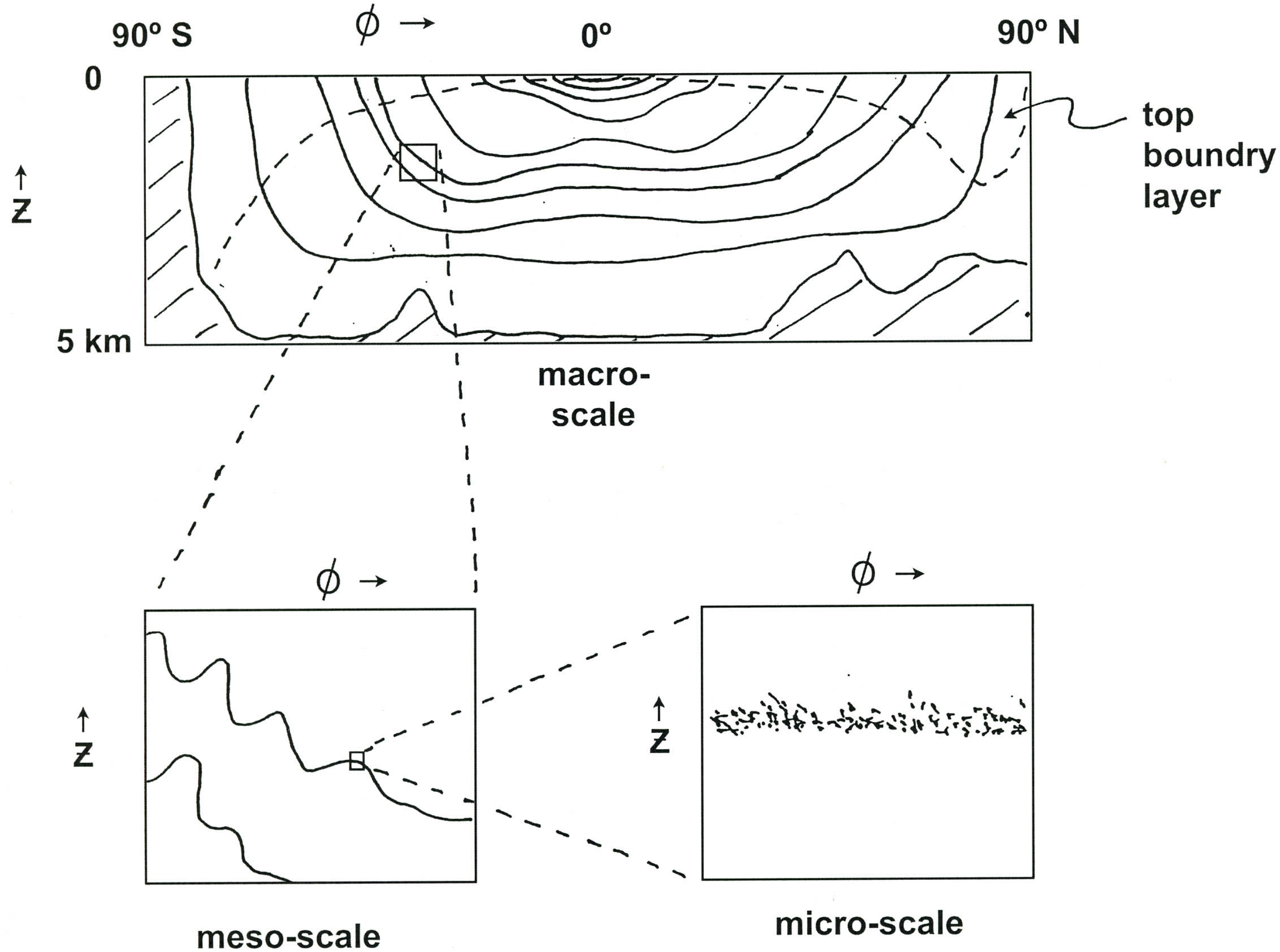
$$w = 0 \quad \text{and} \quad \phi = g\eta \quad \text{at} \quad z = 0\,, \tag{1}$$

referred to as the rigid-lid conditions. The former is consistent with the surface wave-averaged w values being small compared to interior ones, and the second is a hydrostatic approximation to the dynamic pressure immediately below the surface after excising the "inverse barometer response" of a quasi-static sea-level deformation opposite to an atmospheric surface pressure anomaly that excites little low-frequency motion in the ocean. This can be made for analytic convenience and was even made historically for computational efficiency (but is no longer algorithmically necessary).

Boundary Layers, Stratification, and Isopycnal Laminae



Schematic vertical profile of buoyancy $b = -g(\rho_\theta/\rho_0 - 1) = \int N^2(z')\,dz'$, where ρ_θ is potential density and N is the stratification frequency (*e.g.*, for internal gravity waves). On the vertical scale of the full depth shown here, compressibility makes ρ increase with depth much more than ρ_θ, but this effect has little dynamical consequence.



Schematic meridional section of potential density (isopycnal) surfaces viewed on three different spatial scales. In the interior, outside boundary layers, the evolution is approximately adiatatic, with $Db/Dt \approx 0$; *i.e.*, parcels mostly move along isopycnal surfaces. The largest scale shows the time-mean or low-frequency structure. The intermediate scale shows how eddies deform the large-scale buoyancy structure on time scales of weeks and months. The finest scale shows how small-scale turbulence mixes the parcels and disrupts the statically stable vertical ordering of $b(z)$ on a time scale of minutes.

Eddy-Induced Advection

For conservative flow within isopycnal layers (laminae), an especially important eddy flux is the eddy volume flux ($\approx$ mass flux / ρ_0 in the Boussinesq equations), *i.e.*, $\overline{\mathbf{u}'h'}$ where h' is the fluctuation in the vertical thickness of a layer bounded by isopycnal surfaces.

We can define an eddy-induced velocity,

$$\mathbf{u}_* = \frac{\overline{\mathbf{u}'h'}}{\overline{h}}\,,$$

and write its flux divergence as

$$\nabla \cdot \overline{\mathbf{u}'h'} = \mathbf{u}_* \cdot \nabla \overline{h}$$

because of the incompressibility constraint $\nabla \cdot \mathbf{u}_* = 0$. $\mathbf{u}_*$ is sometimes called "Stokes drift" (for surface or internal gravity waves) or "bolus" velocity (for mesoscale eddies).

If the implicit space-time averaging for the governing Boussinesq equations is done in a coordinate frame following isopycnal surfaces, then tracer advection is augmented by eddy-induced advection:

$$\mathbf{u} \cdot \nabla C \longrightarrow \overline{\mathbf{u}} \cdot \nabla C + \mathbf{u}_* \cdot \nabla C$$

for any "eddies" that are subgrid-scale to the modeled flow. Thus, $\overline{\mathbf{u}} + \mathbf{u}_*$ is a kind of Lagrangian mean flow. This effect is distinct from the eddy mixing effect in $\mathcal{D}$, which itself is often preferentially along isopycnal surfaces (rather than strictly horizontal).

The theory behind this is rather subtle, but the oceanic effects are often important especially for models of larger-scale flows.

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Concept: The oceanic interior is quasi-adiabatic, where most stirring and mixing occurs along surfaces of constant potential density, which themselves are stably ordered with respect to gravity ("isopycnal laminae").

The diabatic interior processes that "break" isopycnals are relatively weak: over-tuning internal waves, vertical shear instability (Kelvin-Helmholtz), double diffusion (salt fingers), and molecular diffusion, of course.

Chemically distinctive water masses mostly spread along isocpycnal surfaces.

Eddy-induced advection that moves the isopycnal surfaces along with the other materials on them is a process consistent with this quasi-adiabaticity.

This behavior makes isopycnal out-crops (vertical boundary intersections) important atypical locations of strong diabaticity.

Ekman Transport and Pumping: The momentum boundary layers (called Ekman layers) are especially important in the forcing and damping of currents. Their momentum balance is

$$f\hat{\mathbf{z}} \times \mathbf{u}_h \;=\; -\,\partial_z\overline{\mathbf{u}'_h w'} \;\approx\; \partial_z[\kappa_v \partial_z \mathbf{u}_h]\,.$$

At the surface the turbulent momentum flux $\overline{\mathbf{u}'_h w'}$ is equal to $-\tau^s/\rho_0$ and it $\to 0$ going into the interior. Independent of the profile of $\kappa_v(z)$ (parameterized), the "transport" relation is

$$\mathbf{T}^s_{ek} \;=\; -\frac{1}{f\rho_0}\,\hat{\mathbf{z}} \times \tau^s\,, \qquad \mathbf{T}^s_{ek} \;\equiv\; \int_{-h^s_{ek}}^{\eta} \mathbf{u}^s_{ek}(z)\,dz\,.$$

h^s_{ek} is the Ekman layer thickness, typically $\mathcal{O}(100)$ m, and it scales with the "friction velocity" and f as $h_{ek} \;\sim\; u_*/f$ with $u_* \;=\; \sqrt{|\tau|/\rho_0}$. The surface stress itself is often empirically specified from the near-surface wind $\mathbf{U}_{atm}$ by a bulk formula,

$$\tau^s \;=\; \rho_{atm} C^s_D |\mathbf{U}_{atm}| \mathbf{U}_{atm}\,, \tag{2}$$

with $C_D = \mathcal{O}(10^{-3})$ the drag coefficient for wind over waves. Thus, there is a shallow surface-layer Ekman transport perpendicular to the wind, *i.e.*, rotated clockwise viewed from above in the Northern Hemisphere. The Ekman profile itself, $\mathbf{u}^s_{ek}(z)$, rotates clockwise, and this is called the Ekman spiral.

An analogous bottom Ekman layer due to the bottom stress is

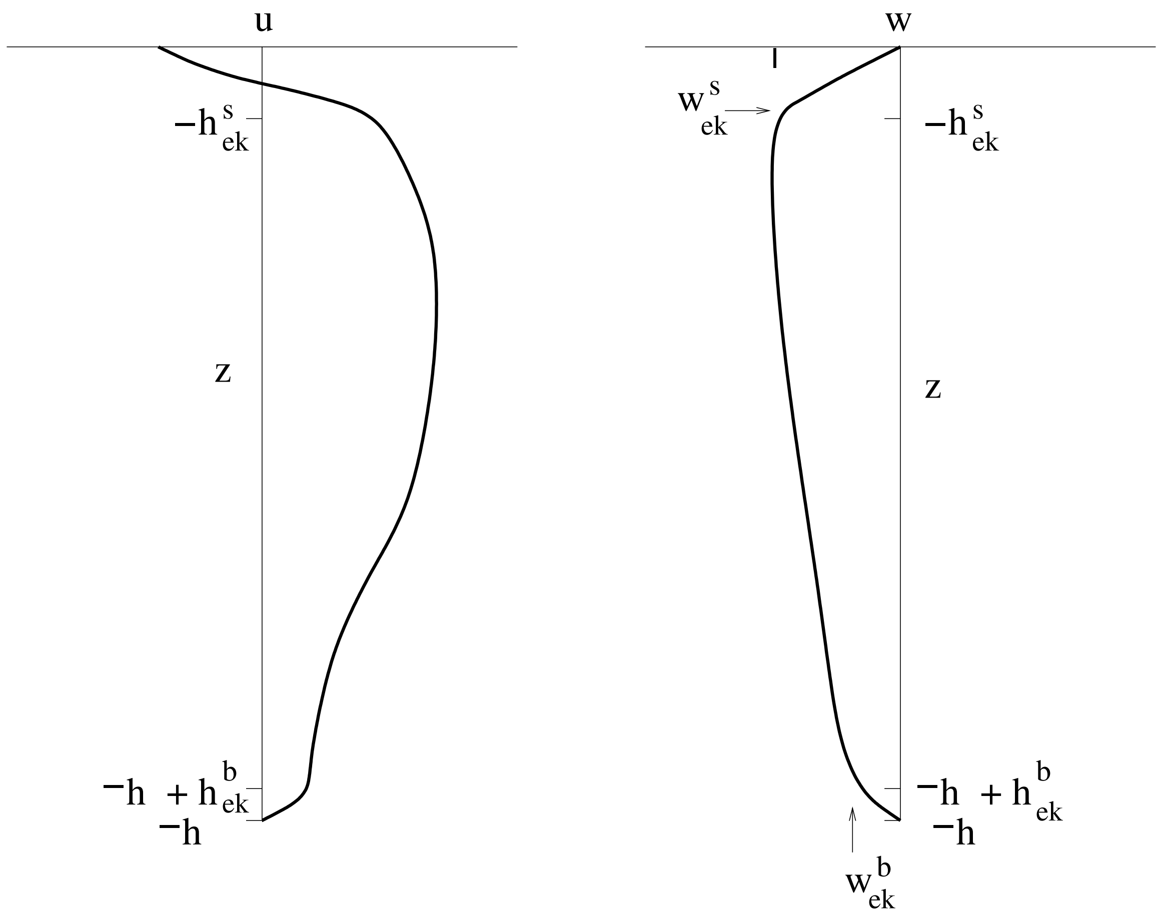
$$\mathbf{T}^b_{ek} \;=\; \frac{1}{f\rho_0}\,\hat{\mathbf{z}} \times \tau^b\,, \qquad \mathbf{T}^b_{ek} \;\equiv\; \int_{-h}^{-h+h^b_{ek}} \mathbf{u}^b_{ek}(z)\,dz\,,$$

with a bulk formula for drag of current over the bottom, $\tau^b \;=\; \rho_0 C^b_D |\mathbf{u}^b_i| \mathbf{u}^b_i$ ("i" denotes interior).

Large- and meso-scale currents typically have a vertical structure with

$$\mathbf{u}_h \;=\; \mathbf{u}_i(z) + \mathbf{u}_{ek}^s(z) + \mathbf{u}_{ek}^b(z) \tag{3}$$

with the interior horizontal current $\mathbf{u}_i$ having larger vertical scales than the Ekman currents and usually satisfying geostrophic balance.



Schematic vertical profiles of horizontal (left) and vertical velocity (right) with top and bottom Ekman boundary layers and interior flow.

The incompressible continuity equation, $\nabla \cdot \mathbf{u} = 0$, allows us to determine w from vertical integration of $-\nabla \cdot \mathbf{u}_h$. In particular, for the Ekman layers with $w = 0$ at the boundary (rigid-lid and flat bottom), the vertical velocities at the interior edge of the layers are

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\begin{aligned}
w^s_{ek} &= \frac{1}{f\rho_0}\hat{\mathbf{z}} \cdot \nabla_h \times \tau^s \\
w^b_{ek} &= \frac{1}{f\rho_0}\hat{\mathbf{z}} \cdot \nabla_h \times \tau^b \,,
\end{aligned} \tag{4}
$$

neglecting horizontal variations in f and h_{ek}. These are called the Ekman pumping velocities. In the vertical interior the $w(z)$ smoothly connects these top and bottom values (previous figure). w^s acts to force the interior currents $\mathbf{u}^i$ by conveying the "curl of the wind stress", while w^b acts to retard $\mathbf{u}^i$ by conveying a "curl of the bottom drag stress", as explained in the next slides. (There are many other dynamical influences on the interior flow evolution in addition to these boundary layer effects.)

With tropical Trade Winds (to the west) and mid-latitude Westerlies (to the east), the meridional Ekman transports diverge at the Equator (implying upwelling for mass balance) and converge in the subtropics (implying downwelling). (Also see MOC below.)

Sverdrup Transport: Another simple transport relation comes from the vertical vorticity equation ($i.e.$, $\hat{\mathbf{z}} \cdot \nabla\times$ operating on the horizontal momentum equation). We retain only Coriolis, pressure-gradient, and turbulent mixing forces for the low-frequency, large-scale flow:

$$\hat{\mathbf{z}} \cdot \nabla \times \left[f\hat{\mathbf{z}} \times \mathbf{u}_h + \nabla_h \phi + \partial_z \overline{\mathbf{u}'_h w'} \right] \;\approx\; 0\,.$$

The $\nabla\phi$ term disappears in a vorticity equation. The vertical integral is

$$\int_{-h}^{\eta} \left[\nabla \cdot (f\mathbf{u}_h) + \hat{\mathbf{z}} \cdot \nabla \times \partial_z \overline{\mathbf{u}'_h w'} \right] \;=\; 0\,.$$

The vertically integrated continuity equation says that the horizontal transport is non-divergent because there is no flow through the oceanic top and bottom,

$$\nabla_h \cdot \mathbf{T} \;=\; 0\,,$$

for $\mathbf{T}$ [m^2 s^{-1}] the total horizontal transport, $\mathbf{T} \;=\; \int_{-h}^{\eta} \mathbf{u}_h(z)\, dz$. This implies that the part of the first term $\propto f\nabla \cdot \mathbf{u}_h$ vanishes in the vorticity balance, while the part $\propto \beta v$ does not. Using the stress boundary conditions discussed above, we obtain Sverdrup balance:

$$\beta T^y \;=\; \frac{1}{\rho_0}\, \hat{\mathbf{z}} \cdot \nabla \times \left(\tau^s - \tau^b\right), \qquad (*)$$

(Define the "barotropic" velocity $\mathbf{u}_0$ as the depth-average of $\mathbf{u}_h$; *i.e.*, $\mathbf{u}_0(x, y) = \mathbf{T}/h$. The "baroclinic" velocity is the vertically varying residual, $\mathbf{u}(x, y, z) - \mathbf{u}_0$, unconstrained by the Sverdrup relation.)

Because of its horizontal non-divergence, we can represent $\mathbf{T}$ with a transport streamfunction Ψ [volume flux; m^3 s^{-1}],

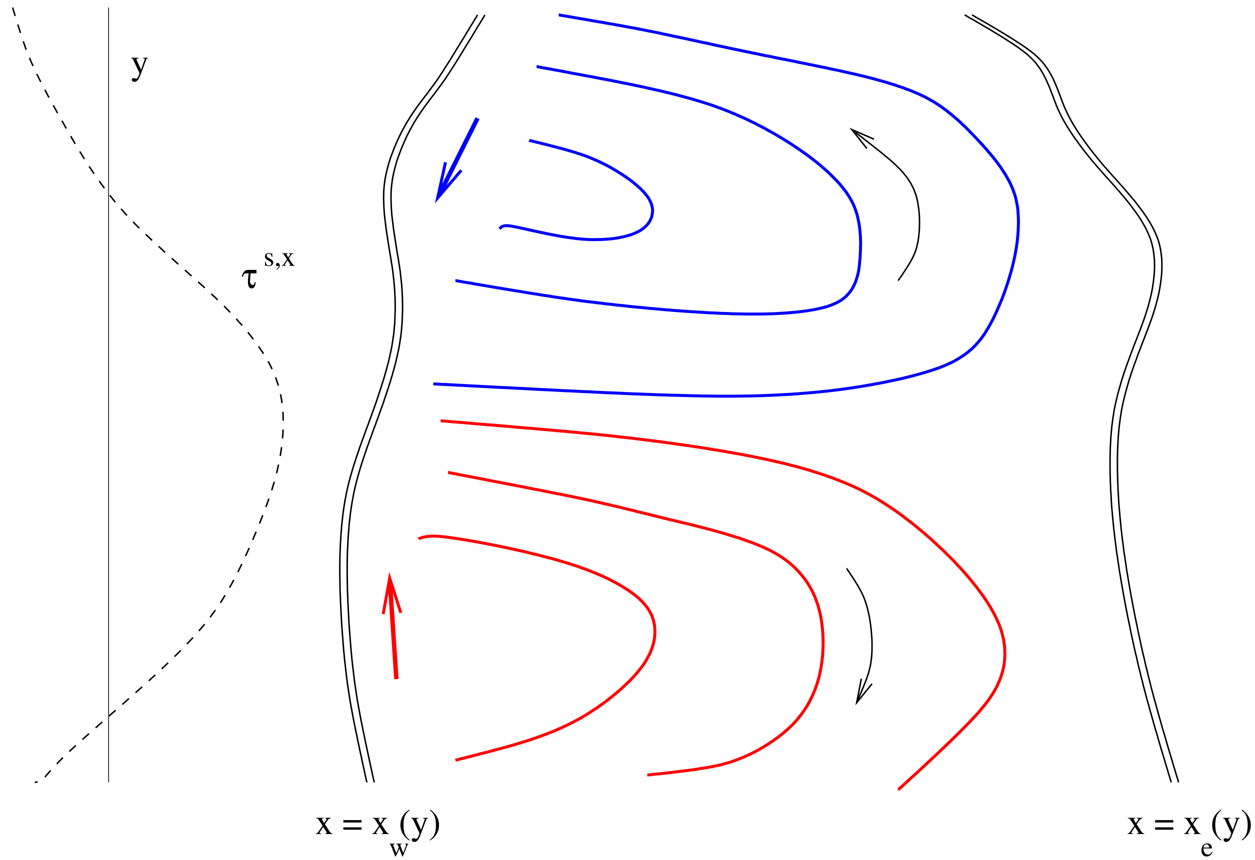
$$T^x = -\partial_y \Psi \,, \qquad T^y = \partial_x \Psi \,.$$

Assume there is no transport through the eastern boundary of an oceanic basin, then the tangential derivative of Ψ vanishes there, hence Ψ is a constant along the boundary (which we take to be zero). We further neglect the bottom stress contribution, as is commonly done for flows away from coastal regions. Thus, integrating (*) in the zonal direction gives the Sverdrup streamfunction,

$$\Psi(x, y) = -\frac{1}{\beta \rho_0} \int_x^{x_e(y)} \mathrm{curl}[\tau^s] \, dx \,,$$

where $x_e(y)$ is the location of the eastern boundary.

Sverdrup transport indicates the major wind-driven gyres in the oceans, away from the Equator where $f \to 0$ and $h_{ek} \to \infty$ (disregarding limitation by stable stratification) and from the Antarctic Circumpolar Current (ACC) where zonal basin boundaries don't exist. Typical wind-gyre magnitudes for Ψ are $\mathcal{O}(10^7)$ m^3 s^{-1}. The transport unit is commonly expressed as a "Sverdrup", where 1 Sv $= 10^6$ m^3 s^{-1}.



Sketch of Sverdrup gyres in the Northern-Hemisphere. A zonal surface stress profile is on the left, with a maximum westerly wind at middle latitudes. The associated Sverdrup transport streamfunction $\Psi(x, y)$ is on the right in an oceanic basin bounded on the west and east by x_w and x_e. Flow is along isolines of Ψ as indicated by the black arrows. The subpolar gyre has $\Psi < 0$ (blue contours) and counterclockwise circulation due to the positive wind curl, $-\partial_y \tau^{s,x} > 0$, at high latitudes. The subtropical gyre has $\Psi > 0$ (red contours) and clockwise circulation due to the negative wind curl, $-\partial_y \tau^{s,x} < 0$, in middle latitudes. The flow near the western boundary is not sketched because it does not satisfy the Sverdrup balance. However, by transport non-divergence within a bounded basin, $\int_{x_w}^{x_e} v_0 \, dx = 0$, there must be meridional return flow for mass balance; this occurs in western boundary currents (WBCs) indicated by the colored arrows.

Meridional Overturning Circulation (MOC): Another major global oceanic circulation mode is the vertically overturning circulation. Because the primary wind systems are zonal, hence the surface Ekman transports are meridional, and because the planetary-scale surface buoyancy flux differences are primarily meridional, the overturning circulation of most interest is the one in the meridional plane, $i.e.$, (y, z). This is defined as the zonally-integrated meridional and vertical transport, which is non-divergent in this plane, hence has an overturning transport streamfunction $\Phi(y, z)$ [m^3 s^{-1} or Sv] analogous to the barotropic Ψ in the (x, y) plane; $i.e.$,

$$\int_{x_w(y,z)}^{x_e(y,z)} v\, dx \;=\; -\,\partial_z \Phi\,, \qquad \int_{x_w(y,z)}^{x_e(y,z)} w\, dx \;=\; \partial_y \Phi\,.$$

x_w and x_e are, respectively, the western and eastern edges of a basin in a zonal transect, and it is often true that there are multiple segments to a transect between continents. The MOC non-divergence is derived by zonally integrating the 3D continuity equation and making use of the kinematic bottom boundary conditions rewritten as, $e.g.$,

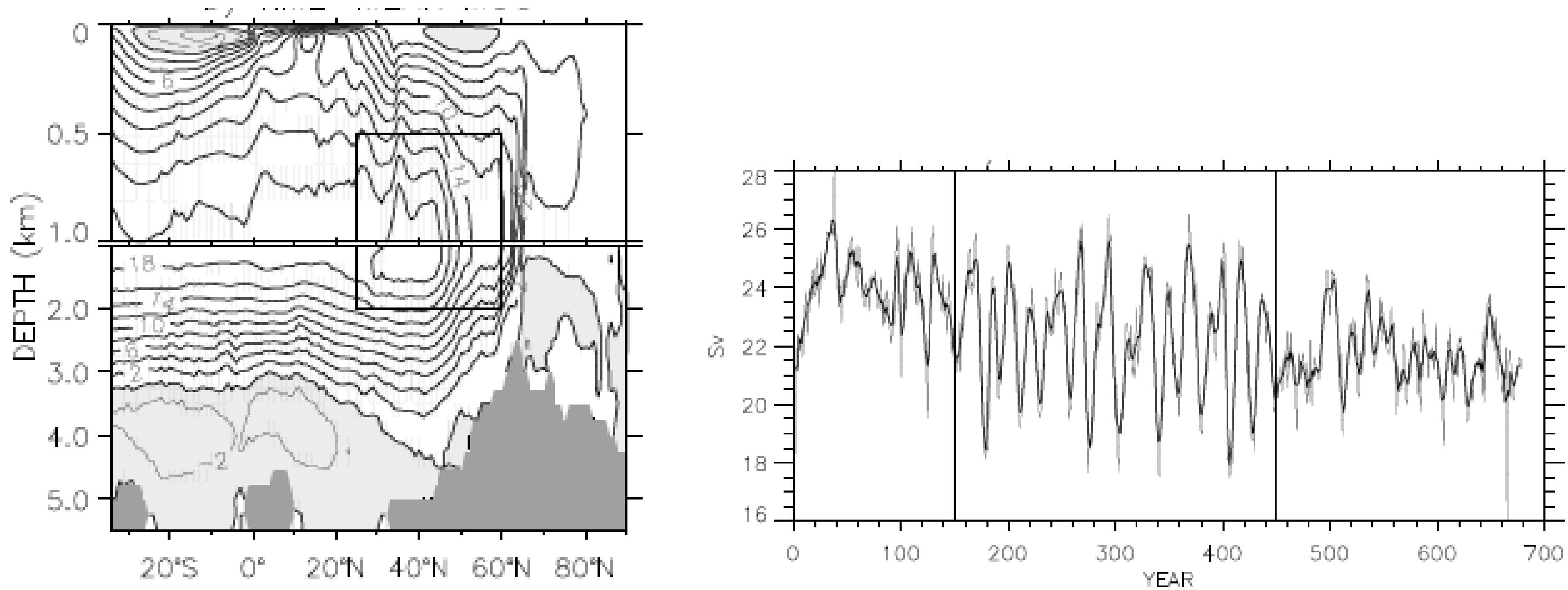
$$\frac{Dx_e}{Dt} \;=\; 0 \quad \text{at} \quad x \;=\; x_e(y, z)\,,$$

$$u_e - v_e(\partial_y x_e) - w_e(\partial_z x_e) \;=\; 0\,,$$

with the subscript denoting evaluation at the boundary. With the rigid-lid approximation and neglect of possible transpolar transport ($i.e.$, across the North Pole),

$$\Phi \;=\; 0 \quad \text{at} \quad z \;=\; 0\,, -max_x[h]\,,$$

because there is no net ($i.e.$, barotropic) MOC transport in a vertical integral.



(Left) Time-mean MOC transport streamfunction Φ [Sv] in the Atlantic Ocean. (Right) Time series of the annual-mean maximum value of Φ within the box in the top panel centered around the North Atlantic Deep Water transport cell. These are from a coupled climate simulation in modern-day equilibrium (Danabasoglu, 2008). The maximum for Φ is > 0 and about 22 Sv, indicating the dominant circulation cell with sinking in the far north Atlantic forced primarily by subpolar surface cooling and deep convection in the Norwegian and Labrador Seas. Its abyssal southward outflow is called the North Atlantic Deep Water (NADW) water mass. There is an opposite abyssal cell near the bottom, called the Antarctic Bottom Water (AABW) which originates with sinking off Antarctica. There are also several shallow MOC cells caused by Ekman pumping: a pair of subtropical cells on either side of the Equatorial upwelling, and a weak northern subpolar cell. The dominant time variability of the NADW cell is on a decadal times scale, in association with the North Atlantic Oscillation (NAO) in tropospheric climate. The Atlantic MOC is part of the global "conveyor belt" circulation, which has significant climate heat transport.

Ertel's Potential Vorticity: With $\mathcal{D} = 0$, Boussinesq equations have a conservative Lagrangian invariant in addition to b:

$$\frac{DQ}{Dt} \approx 0\,, \qquad Q = (2\mathbf{\Omega} + \nabla \times \mathbf{u}) \cdot \nabla b\,.$$

Insofar as b and Q are independently configured, then knowledge of both provides a very strong constraint on the circulation: trajectories must lie along the line that is the intersection of the surfaces of constant b and constant Q as they flop around.

Planetary Geostrophy (PG): For large-scale circulation, geostrophic and hydrostatic balance are valid. Neglecting $\mathcal{D}$ (*i.e.*, neglecting eddy fluxes), the planetary potential vorticity balance is

$$[\,\partial_t + u\partial_x + v\partial_y + w\partial_z\,]\,Q_{pg} = 0\,, \qquad Q_{pg} = f\partial_z b = fN^2\,.$$

$\Delta\varphi$ can be large, with $f(\varphi)$ beyond the β-plane. This can be combined with Sverdrup balance to describe the 3D pycnocline circulation (a.k.a. "thermocline theory").

Quasigeostrophy (QG): For mesoscale circulation, broadly defined as $L \sim R_d$, where $R_d = NH/f$ is the baroclinic deformation radius. Again with goestrophic-hydrostatic balance ($Ro \sim Fr \ll 1$) and $\mathcal{D} = 0$, the vorticity and buoyancy equations can be combined to yield

$$[\,\partial_t + \mathbf{u}_g \cdot \nabla\,]\,Q_{qg} = 0\,, \qquad Q_{qg} = (\partial_x v_g - \partial_y u_g) + f_0\,\partial_z\left(\frac{b'}{\overline{N}^2}\right) + \beta y\,,$$

where $b'(x,y,z,t)$ is the fluctuation around the mean stratification $\overline{b}(z)$.

"Shallow-Water" (SW): For a layer of constant density and bounding interfaces η_{top} and η_{bot},

$$[\,\partial_t + \mathbf{u} \cdot \nabla\,]\,Q_{sw} = 0\,, \quad Q_{sw} = \frac{f(y) + \partial_x v - \partial_y u}{h}\,, \qquad h(x,y,t) = h_0 + \eta_{top} - \eta_{bot}\,.$$

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Concept: Oceanic circulation is multi-scale, spatially and temporally from 10^{-3} to 10^{7} m and from 10^{0} to 10^{11} s, and the fundamental equations are too hard to solve in a general way, either analytically or computationally.

So oceanic theory and modeling is a game of approximations and compromises.

A powerful technique is scale estimation using typical numbers to represent relevant components of the multi-scale fields and drawing conclusions about which simplifying approximations are justifiable.

$e.g.,$ a mesoscale eddy might have $V \approx 0.1$ m s^{-1} and $L \approx$ 50 km at a latitude where $f \approx 10^{-4}$ s^{-1} with $H \approx 1000$ m across the pycnocline and $N \approx 3 \times 10^{-3}$ s^{-1}; hence, $Ro \approx 0.02$, $Fr \approx 0.03$, $L_d \approx 30$ km, and a QG model is probably OK.

Oceanic General Circulation Models (OGCMs)

Most of the elements of a modern, global OGCM have already been described, and here we put them together. A somewhat dated review is in McWilliams (1996).

• The governing fluid dynamics are the Primitive Equations with the traditional Coriolis approximation. The spatial coordinates are for a thin spherical shell geometry.

• The PDEs are solved on a discretized spatial grid. The vertical coordinate is differently geopotential, isopycnic, or sigma (terrain-following) in different OGCM types. A typical grid resolution for a global model is $dx =$ 100 km (without mesoscale eddies, as in most climate models) or 10 km (with them) in the horizontal, and $dz =$ 5 - 200 m in the vertical with the grid spacing increasing with depth, commensurate with the expected profiles of $\mathbf{u}$ and C. (Regional and coastal circulation simulations are made with much finer grid resolution; they have the added requirement for boundary data at their open-ocean boundaries, taken either from climatological analyses or larger-scale model solutions.)

• There is discretized time integration with a typical $dt = 1$ hr to resolve inertial and some internal gravity wave currents. Initial conditions are either from climatological data or from previous solutions. Integration to steady-state requires several 1000 yrs, but common practice is for shorter (cheaper) solutions, where most of the upper ocean fields and barotropic currents come into a quasi-steady balance after several ten yrs.

• The EOS is in the form of $\rho(T, S, z)$, with T sometimes chosen as potential or "conservative" temperature to diminish compressibility effects.

• The surface fluxes (τ^s, Q, and $\mathcal{F}$) are taken from meteorological data sets or are calculated in a coupled ocean-atmosphere climate model. Usually $\mathcal{F}$ is converted into a virtual salinity flux because most oceanic models assume constant oceanic volume of seawater. River run-off is included in $\mathcal{F}$.

• The surface kinematic boundary condition is a free surface, even though the solution behavior is usually consistent with the rigid-lid approximation. It may also include a coupled sea ice model floating on top.

• Bottom depth is taken from bathymetric data sets, with appropriate smoothing to accommodate the model grid resolution. In global models a minimum side depth is chosen for computational prudence ($e.g.$, $h_{min} =$ 50 m).

• Parameterizations are needed for unresolved (subgrid-scale) turbulent mixing, usually as eddy diffusion. This is done separately for vertical mixing in top and bottom turbulent boundary layers and for horizontal/vertical or isopycnal/diapycnal mixing in the interior. The important horizontal mixing is contributed by mesoscale eddies, which are either wholly or partly parameterized in coarse or fine resolution OGCM solutions, respectively. Partial eddy resolution requires partial parameterization with a reduced magnitude of the eddy-induced velocity and diffusivity, $\mathbf{u}_*$ and κ.

• Tides are usually not included in OGCM formulations, although it is technically feasible to do so. This is because tidal circulations are viewed as largely decoupled from the general circulation and eddies. An exception, however, is that tidally induced diapycnal mixing is a significant effect for long-lived tracers and the thermohaline circulation ($i.e.$, abyssal MOC), and this effect is parameterized in an OGCM.

• Compatibly simplified biogeochemical and ecosystem dynamics are required for computational feasibility, and boundary flux data sets are needed.

OGCM solutions are skillful in representing the surface boundary layer, seasonal cycle, major wind gyres, equatorial zonal currents, thermohaline circulation, MOC, and primary water masses including the biogeochemical tracers. In coupled mode with an atmospheric GCM, they are skillful for large-scale heat and water fluxes, interannual climate variability ($e.g.$, El Niño - Southern Oscillation), and global warming projections. With coarse grid resolution (a.k.a. climate models), OGCMs have overly broad and weak WBCs and they lack mesoscale eddies, and even with the presently feasible fine resolution globally with $dx \approx 10$ km, they do not resolve these phenomena very well, although finer resolution regional models can do so.

The ocean has many other important phenomena on even finer scales that are not present at all in OGCM solutions, $e.g.$, , high-frequency internal gravity waves, small vortices, fronts, filaments, shoreline currents, and boundary-layer turbulence. They are often calculated in local or even Large-Eddy Simulation (LES; $i.e.$, with partial resolution of the turbulent currents) models, usually as a process study with an idealized problem formulation.

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