Before, we considered coordinate system rotation and divided the acceleration as seen from space (\(\frac{d\mathbf{U}}{dt}\)) into these parts: the acceleration seen from the Earth (\(\frac{d\mathbf{U}}{dt}\)) and terms relating to Earth rotation (Coriolis and centrifugal accelerations). Now we have a rotation of coordinates due to the sphericity of the Earth. This will entail splitting \(\frac{d\mathbf{U}}{dt}\) into two parts, representing accelerations relative to a flat Earth and compensation for Earth curvature.

**Velocity components**

Our unit vectors \(\hat{i}, \hat{j}, \) and \(\hat{k}\) remain pointing east, north and up, but we need to move from Cartesian position \(x, y, \) and \(z\) to spherical position \(\lambda, \phi, \) and \(z\) where \(\lambda\) is longitude and \(\phi\) is latitude. Thus, we need to relate velocities \(u\) and \(v\) in terms of \(\frac{d\lambda}{dt}\) and \(\frac{d\phi}{dt}\).

![Meridional velocity](image)

Figure 1: Meridional velocity \(v\) as a tangential velocity.

Figure 1 shows that the north-south (meridional) velocity \(v\) along a given longitude \(\lambda\) can be interpreted as a velocity tangent to a circle of distance \(r\) from the center of the Earth. (This \(r = a + z\), where \(a\) is the Earth’s radius and \(z\) is distance above the Earth’s surface.) Recall that tangential velocity is is angular velocity times radius. The angular velocity is \(\frac{d\phi}{dt}\) since latitude is changing with time. Therefore, we have

\[
v = r \frac{d\phi}{dt}.
\]

(1)

Now look at east-west (zonal) velocity \(u\), representing a longitude change \(d\lambda\) with time along a latitude circle (Fig. 2). It is clear that \(u\) is also a tangential velocity. The radius of the latitude circle is \(R\), distance from the spin axis, which may also be expressed as \(r \cos \phi\). It follows that

\[
u = R \frac{d\lambda}{dt} = r \cos \phi \frac{d\lambda}{dt}.
\]

(2)

Finally, \(w = \frac{dz}{dt}\) on both flat and spherical Earths.
Chain rule

With regard to 3D vector velocity, \( \vec{U} = u \hat{i} + v \hat{j} + w \hat{k} \), six aspects of this expression could change with time: the component velocities, and the coordinate axes. Thus the chain rule applied to \( \frac{d\vec{U}}{dt} \) yields

\[
\frac{d\vec{U}}{dt} = \frac{du}{dt} \hat{i} + \frac{dv}{dt} \hat{j} + \frac{dw}{dt} \hat{k} + u \frac{d\hat{i}}{dt} + v \frac{d\hat{j}}{dt} + w \frac{d\hat{k}}{dt}.
\]  

We need to find expressions for the coordinate axis accelerations.

This effort starts with identifying the dimensions in which each axis varies. The \( \hat{i} \) axis is simplest since it is a function only of longitude. Moving along any given latitude circle, \( \hat{i} \) changes direction owing to Earth’s curvature, as shown in Fig. 3. The other two coordinate axes vary in both latitude and longitude, as shown in Figures 4 and 5.

Figure 3: The \( \hat{i} \) coordinate axis varies with longitude only.
Figure 4: The \( \hat{j} \) coordinate axis varies with longitude and latitude.

Figure 5: The \( \hat{k} \) coordinate axis varies with longitude and latitude.

The \( \hat{i} \) axis

The \( \hat{i} \) unit vector always points east, but what we define as “east” changes around a latitude circle. Indeed, \( \hat{i} \) can be interpreted as tangential velocity and its change \( \Delta \hat{i} \) as a centripetal acceleration, as shown in Fig. 6. Recognizing that \( \hat{i} \) varies only longitudinally, the chain rule applied to \( \frac{d\hat{i}}{dt} \) quickly reduces in the following manner:

\[
\frac{d\hat{i}}{dt} = \frac{\partial \hat{i}}{\partial t} + u \frac{\partial \hat{i}}{\partial x} + v \frac{\partial \hat{i}}{\partial y} + w \frac{\partial \hat{i}}{\partial z} = u \frac{\partial \hat{i}}{\partial x}.
\]

We will need \( \frac{d\hat{i}}{dt} \), and that is

\[
\frac{d}{dt} \frac{d\hat{i}}{dt} = u^2 \frac{\partial \hat{i}}{\partial x}.
\]

The derivation continues to mimic that done for centripetal acceleration. The magnitude of \( \frac{\partial \hat{i}}{\partial x} \) is \( \frac{\Delta i}{\Delta x} \), where \( dx \) is the arclength over angle \( d\lambda \) (i.e., \( dx = r \cos \phi \Delta \lambda \)). For small \( d\lambda \), \( \Delta \hat{i} \approx d\lambda \) (recall unit vectors have unit length). Therefore,

\[
\frac{\Delta \hat{i}}{\Delta x} = \frac{\Delta \lambda}{r \cos \phi \Delta \lambda} = \frac{1}{r \cos \phi}.
\]

Just as with centripetal acceleration, we note \( \delta \hat{i} \) points inward towards the spin axis, and has two components – northward and towards Earth’s center – as shown in Fig. 7. The northward
component is $\Delta \hat{i} \sin \phi \hat{j}$ and the other is $-\Delta \hat{i} \cos \hat{k}$. As a result,
\[ \frac{\partial \hat{i}}{\partial x} = \frac{1}{r \cos \phi} \left( \hat{j} \sin \phi - \hat{k} \cos \phi \right), \]  
leading to
\[ u^2 \frac{\partial \hat{i}}{\partial x} = \frac{u^2}{r \cos \phi} \left( \hat{j} \sin \phi - \hat{k} \cos \phi \right). \]  
This creates a $\frac{u^2}{r^2} \tan \phi$ term for the $v$ equation of motion and a $\frac{u^2}{r^2}$ term for the $w$ equation.

The $\hat{j}$ and $\hat{k}$ equations

The other two coordinates are more complicated, owing to dependence on both longitude and latitude, but the accelerations are derived in a very similar way. For the $\hat{j}$ unit vector, we obtain
\[ u \frac{\partial \hat{j}}{\partial x} = -\frac{u \tan \phi}{r} \hat{i}, \]  
Figure 6: An augmented version of Fig. 3, showing the change of the $\hat{i}$ axis along a latitude circle is interpretable as a centripetal acceleration.

Figure 7: $\Delta \hat{i}$ and its components.
\[
v \frac{\partial j}{\partial y} = -\frac{v}{r}\hat{k}
\]  

(11)

for the latitudinal and longitudinal dependences, respectively. The \( \hat{k} \) component results in

\[
\frac{d\hat{k}}{dt} = \frac{u}{r}\hat{i} + \frac{v}{r}\hat{j}.
\]  

(12)

Therefore, (3) expands into

\[
\frac{d\vec{U}}{dt} = \left[ \frac{du}{dt} - \frac{uv \tan \phi}{r} + \frac{uw}{r} \right] \hat{i} + \left[ \frac{dv}{dt} + \frac{u^2 \tan \phi}{r} + \frac{vw}{r} \right] \hat{j} + \left[ \frac{dw}{dt} - \frac{u^2 + v^2}{r} \right] \hat{k}.
\]