A&OS 101 - Accelerations owing to sphericity

Fall, 2005 – Fovell

Before, we considered coordinate system rotation and divided the acceleration as seen from space ($\frac{d_a\vec{U}_a}{dt}$) into these parts: the acceleration seen from the Earth ($\frac{d\vec{U}}{dt}$) and terms relating to Earth rotation (Coriolis and centrifugal accelerations). Now we have a rotation of coordinates due to the sphericity of the Earth. This will entail splitting $\frac{d\vec{U}}{dt}$ into two parts, representing accelerations relative to a flat Earth and compensation for Earth curvature.

Velocity components

Our unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ remain pointing east, north and up, but we need to move from Cartesian position x, y and z to spherical position λ, ϕ and z where λ is longitude and ϕ is latitude. Thus, we need to relate velocities u and v in terms of $\frac{d\lambda}{dt}$ and $\frac{d\phi}{dt}$.

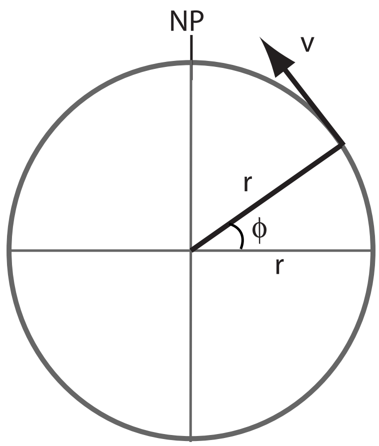

Figure 1: Meridional velocity v as a tangential velocity.

Figure 1 shows that the north-south (meridional) velocity v along a given longitude λ can be interpreted as a velocity tangent to a circle of distance r from the center of the Earth. (This $r = a + z$, where a is the Earth's radius and z is distance above the Earth's surface.) Recall that tangential velocity is is angular velocity times radius. The angular velocity is $\frac{d\phi}{dt}$ since latitude is changing with time. Therefore, we have

$$v = r\frac{d\phi}{dt}. \tag{1}$$

Now look at east-west (zonal) velocity u, representing a longitude change $d\lambda$ with time along a latitude circle (Fig. 2). It is clear that u is also a tangential velocity. The radius of the latitude circle is R, distance from the spin axis, which may also be expressed as $r\cos\phi$. It follows that

$$u = R\frac{d\lambda}{dt} = r\cos\phi\frac{d\lambda}{dt}. \tag{2}$$

Finally, $w = \frac{dz}{dt}$ on both flat and spherical Earths.

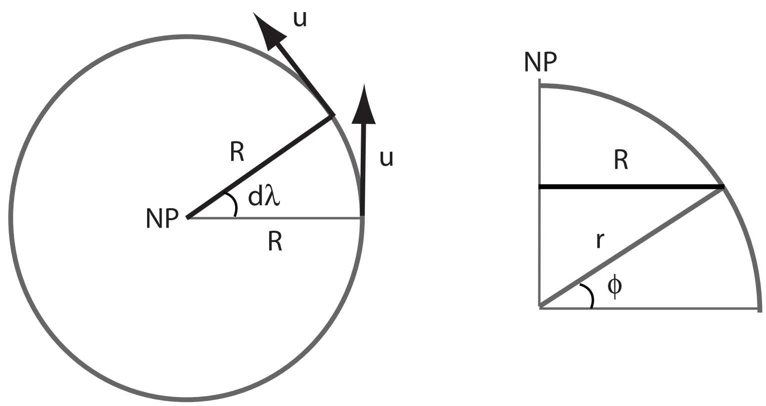

Figure 2: Zonal velocity u as a tangential velocity.

Chain rule

With regard to 3D vector velocity, $\vec{U} = u\hat{i} + v\hat{j} + w\hat{k}$, six aspects of this expression could change with time: the component velocities, and the coordinate axes. Thus the chain rule applied to $\frac{d\vec{U}}{dt}$ yields

$$\frac{d\vec{U}}{dt} = \frac{du}{dt}\hat{i} + \frac{dv}{dt}\hat{j} + \frac{dw}{dt}\hat{k} + u\frac{d\hat{i}}{dt} + v\frac{d\hat{j}}{dt} + w\frac{d\hat{k}}{dt}. \tag{3}$$

We need to find expressions for the coordinate axis accelerations.

This effort starts with identifying the dimensions in which each axis varies. The $\hat{i}$ axis is simplest since it is a function only of longitude. Moving along any given latitude circle, $\hat{i}$ changes direction owing to Earth's curvature, as shown in Fig. 3. The other two coordinate axes vary in both latitude and longitude, as shown in Figures 4 and 5.

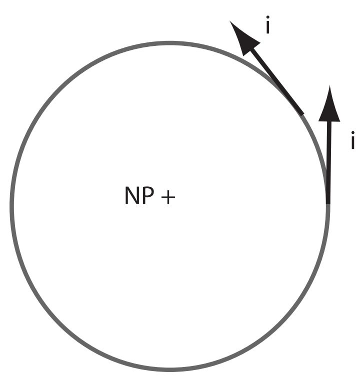

Figure 3: The $\hat{i}$ coordinate axis varies with longitude only.

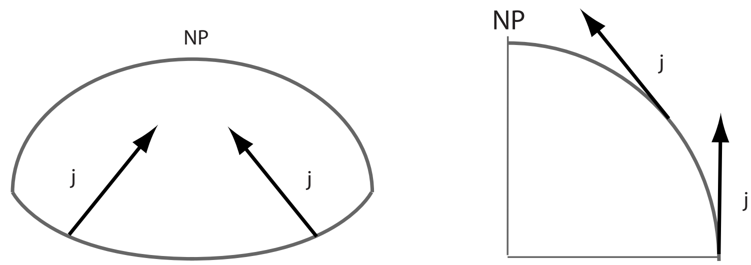

Figure 4: The $\hat{j}$ coordinate axis varies with longitude and latitude.

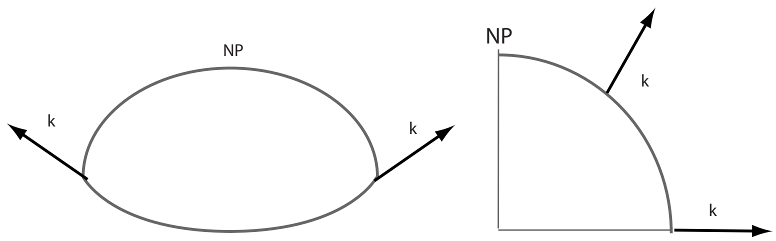

Figure 5: The $\hat{k}$ coordinate axis varies with longitude and latitude.

The $\hat{i}$ axis

The $\hat{i}$ unit vector always points east, but what we define as "east" changes around a latitude circle. Indeed, $\hat{i}$ can be interpreted as tangential velocity and its change $\Delta\hat{i}$ as a centripetal acceleration, as shown in Fig. 6. Recognizing that $\hat{i}$ varies only longitudinally, the chain rule applied to $\frac{d\hat{i}}{dt}$ quickly reduces in the following manner:

$$\frac{d\hat{i}}{dt} = \frac{\partial\hat{i}}{\partial t} + u\frac{\partial\hat{i}}{\partial x} + v\frac{\partial\hat{i}}{\partial y} + w\frac{\partial\hat{i}}{\partial z} \quad (4)$$

$$= u\frac{\partial\hat{i}}{\partial x}. \quad (5)$$

We will need $\frac{d\hat{i}}{dt}$, and that is

$$d\frac{d\hat{i}}{dt} = u^2\frac{\partial\hat{i}}{\partial x}. \quad (6)$$

The derivation continues to mimic that done for centripetal acceleration. The magnitude of $\frac{\partial\hat{i}}{\partial x}$ is $\frac{\Delta\hat{i}}{\Delta x}$, where dx is the arclength over angle $d\lambda$ (i.e., $dx = r\cos\phi\Delta\lambda$). For small $d\lambda$, $\Delta\hat{i} \approx d\lambda$ (recall unit vectors have unit length). Therefore,

$$\frac{\Delta\hat{i}}{\Delta x} = \frac{\Delta\lambda}{r\cos\phi\Delta\lambda} = \frac{1}{r\cos\phi}. \quad (7)$$

Just as with centripetal acceleration, we note $\delta\hat{i}$ points inward towards the spin axis, and has two components – northward and towards Earth's center – as shown in Fig. 7. The northward

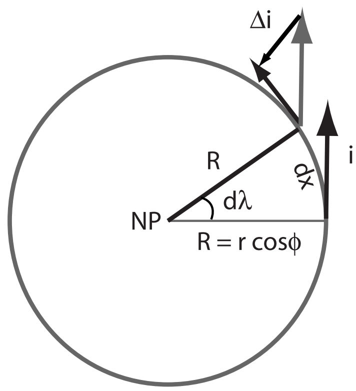

Figure 6: An augmented version of Fig. 3, showing the change of the $\hat{i}$ axis along a latitude circle is interpretable as a centripetal acceleration.

component is $\Delta\hat{i}\sin\phi\hat{j}$ and the other is -$\Delta\hat{i}\cos\phi\hat{k}$. As a result,

$$\frac{\partial\hat{i}}{\partial x} = \frac{1}{r\cos\phi}\left(\hat{j}\sin\phi - \hat{k}\cos\phi\right), \tag{8}$$

leading to

$$u^2\frac{\partial\hat{i}}{\partial x} = \frac{u^2}{r\cos\phi}\left(\hat{j}\sin\phi - \hat{k}\cos\phi\right). \tag{9}$$

This creates a $\frac{u^2}{r}\tan\phi$ term for the v equation of motion and a $\frac{u^2}{r}$ term for the w equation.

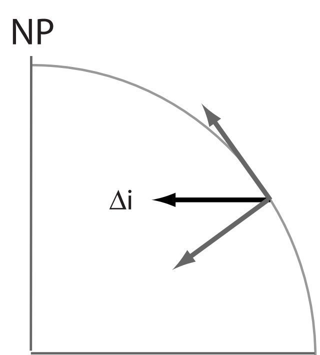

Figure 7: $\Delta\hat{i}$ and its components.

The $\hat{j}$ and $\hat{k}$ equations

The other two coordinates are more complicated, owing to dependence on both longitude and latitude, but the accelerations are derived in a very similar way. For the $\hat{j}$ unit vector, we obtain

$$u\frac{\partial\hat{j}}{\partial x} = -\frac{u\tan\phi}{r}\hat{i}, \tag{10}$$

$$v\frac{\partial \hat{j}}{\partial y} = -\frac{v}{r}\hat{k} \tag{11}$$

for the latitudinal and longitudinal dependences, respectively. The $\hat{k}$ component results in

$$\frac{d\hat{k}}{dt} = \frac{u}{r}\hat{i} + \frac{v}{r}\hat{j}. \tag{12}$$

Therefore, (3) expands into

$$\begin{aligned} \frac{d\vec{U}}{dt} &= \left[\frac{du}{dt} - \frac{uv\tan\phi}{r} + \frac{uw}{r}\right]\hat{i} \\ &+ \left[\frac{dv}{dt} + \frac{u^2\tan\phi}{r} + \frac{vw}{r}\right]\hat{j} \\ &+ \left[\frac{dw}{dt} - \frac{u^2+v^2}{r}\right]\hat{k}. \end{aligned}$$