Gravity waves derivation

A&OS 101 - Fall, 2008

Introduction

- Gravity waves describe how environment responds to disturbances, such as by oscillating parcels
- Goal: derive "dispersion relation" that relates wave characteristics (wavelength, period) to disturbance characteristics (oscillation period)
- Simplifications: make atmosphere 2D, dry adiabatic, flat, non-rotating

Starting equations

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

continuity equation

$$\frac{d\theta}{dt} = 0$$

Expand total derivatives

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} &= 0 \\
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g &= 0 \\
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0 \\
\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + w\frac{\partial \theta}{\partial z} &= 0
\end{aligned}
$$

Perturbation method

$$
\begin{aligned}
u(x,z,t) &= u'(x,z,t) \\
w(x,z,t) &= w'(x,z,t) \\
\rho(x,z,t) &= \rho_0 + \rho'(x,z,t) \\
p(x,z,t) &= \bar{p}(z) + p'(x,z,t) \\
\theta(x,z,t) &= \bar{\theta}(z) + \theta'(x,z,t) \\
\frac{d\bar{p}}{dz} &= -\rho_0 g
\end{aligned}
$$

Calm, hydrostatic, constant density environment
("basic state")

Start w/ potential temperature

$$\begin{aligned}\theta &= T\left[\frac{1000}{p}\right]^{\frac{R}{c_p}} \\ \ln\theta &= \frac{c_v}{c_p}\ln p - \ln\rho + \text{consts} \\ \ln\bar{\theta} &= \frac{c_v}{c_p}\ln\bar{p} - \ln\bar{\rho} + \text{consts}\end{aligned}$$

for basic state

used log tricks and:

$$\begin{aligned}c_p &= R + c_v \\ p &= \rho R T\end{aligned}$$

Apply perturbation method

$$\begin{aligned}\ln(\bar{\theta}+\theta') &= \frac{c_v}{c_p}\ln(\bar{p}+p') - \ln(\rho_0+\rho') + \text{consts} \\ \ln\left[\bar{\theta}(1+\frac{\theta'}{\bar{\theta}})\right] &= \frac{c_v}{c_p}\ln\left[\bar{p}(1+\frac{p'}{\bar{p}})\right] - \ln\left[\rho_0(1+\frac{\rho'}{\rho_0})\right] + \text{consts}\end{aligned}$$

Base state cancels... makes constants disappear...

Useful approximation for *x* small: $\ln(1+x) \approx x$

$$\begin{aligned}\frac{\theta'}{\bar{\theta}} &= \frac{c_v}{c_p}\frac{p'}{\bar{p}} - \frac{\rho'}{\rho_0} \\ &= \frac{p'}{\rho_0 c_s^2} - \frac{\rho'}{\rho_0}\end{aligned}$$

$$c_s^2 = \frac{c_p}{c_v}R_d\bar{T}$$ *speed of sound*

$$\text{for } c_s \text{ large}: \frac{\rho'}{\rho_0} \approx -\frac{\theta'}{\bar{\theta}}$$

Vertical equation

$$\rho\frac{\partial w}{\partial t} + \rho u\frac{\partial w}{\partial x} + \rho w\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} + \rho g \quad = \quad 0$$

Do perturbation analysis,
neglect products of perturbations

$$\rho_0\frac{\partial w'}{\partial t} + \frac{d\bar{p}}{dz} + \frac{\partial p'}{\partial z} + \rho_0 g + \rho' g = 0$$

Rearrange, replace density with potential temperature

$$\boxed{\frac{\partial w'}{\partial t} + \frac{1}{\rho_0}\frac{\partial p'}{\partial z} - g\frac{\theta'}{\bar{\theta}} = 0}$$

Our first pendulum equation

$$\frac{\partial w'}{\partial t} + \frac{1}{\rho_0}\frac{\partial p'}{\partial z} - g\frac{\theta'}{\bar{\theta}} = 0$$

We now know an oscillating parcel will disturb its environment and p' plays a crucial, non-negligible role

Full set of linearized equations

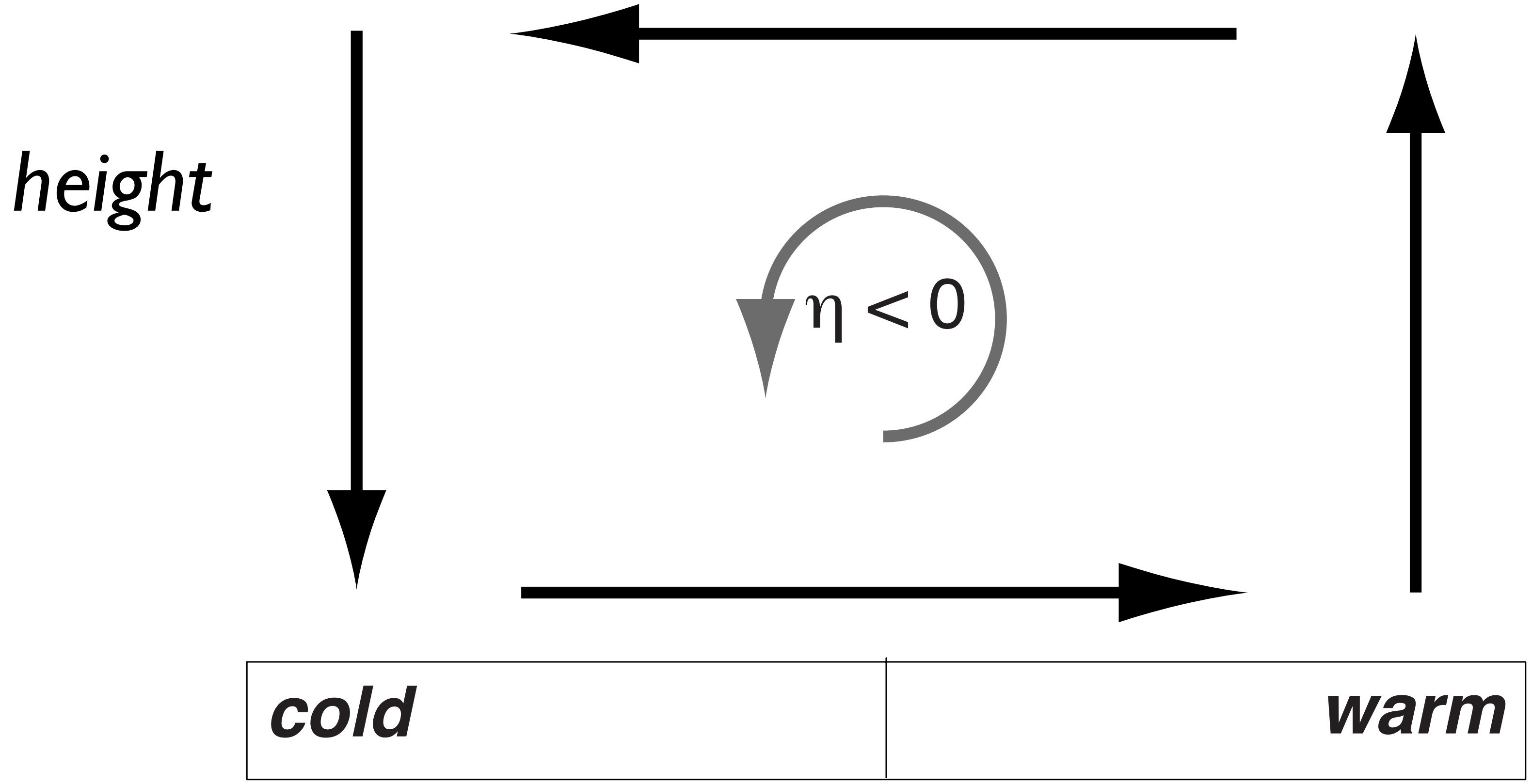
$$
\begin{aligned}
\frac{\partial u'}{\partial t} + \frac{1}{\rho_0}\frac{\partial p'}{\partial x} &= 0 \\
\frac{\partial w'}{\partial t} + \frac{1}{\rho_0}\frac{\partial p'}{\partial z} - g\frac{\theta'}{\bar{\theta}} &= 0 \\
\frac{\partial \theta'}{\partial t} + w\frac{d\bar{\theta}}{dz} &= 0 \\
\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} &= 0.
\end{aligned}
$$

Vorticity form of these equations

- Vorticity == *spin*
- Vorticity defined by spin axis
  - horizontal vorticity is spin in vertical plane
- Horizontal vorticity parallel to the *y* axis:

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

Recall sea-breeze circulation



$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

negative horizontal vorticity by right hand rule

Obtaining the vorticity equation

differentiate horizontal and vertical equations of motion

$$\frac{\partial}{\partial z}\left[\frac{\partial u'}{\partial t} + \frac{1}{\rho_0}\frac{\partial p'}{\partial x}\right] = 0$$

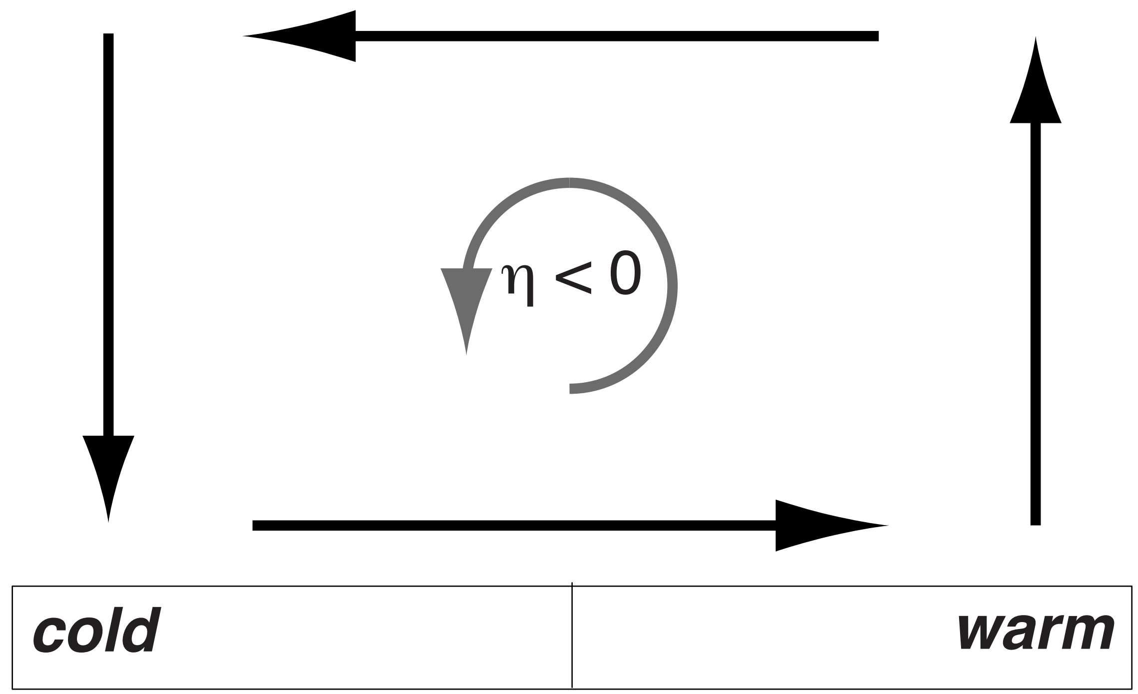
$$\frac{\partial}{\partial x}\left[\frac{\partial w'}{\partial t} + \frac{1}{\rho_0}\frac{\partial p'}{\partial z} - g\frac{\theta'}{\bar{\theta}}\right] = 0$$

subtract top equation from bottom

$$\frac{\partial}{\partial t}\left[\frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z}\right] - \frac{g}{\bar{\theta}}\frac{\partial \theta'}{\partial x} = 0$$

where $$-\eta' = \frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z}$$

Sea-breeze circulation



$$\frac{\partial \eta'}{\partial t} = -\frac{g}{\bar{\bar{\theta}}}\frac{\partial \theta'}{\partial x}$$

Temperature perturbation increases in +x direction creates counterclockwise spin in vertical plane

Back up a step and differentiate again

$$\frac{\partial}{\partial t}\left[\frac{\partial^2 w'}{\partial x^2} - \frac{\partial^2 u'}{\partial z \partial x}\right] - \frac{g}{\bar{\theta}}\frac{\partial^2 \theta'}{\partial x^2} = 0$$

...differentiated w/r/t *x*. Since continuity implies

$$\frac{\partial u'}{\partial x} = -\frac{\partial w'}{\partial z} \qquad \text{then} \qquad \frac{\partial^2 u'}{\partial x \partial z} = -\frac{\partial^2 w'}{\partial z^2}$$

Since the potential temperature equation was

$$\frac{\partial \theta'}{\partial t} = -w\frac{d\bar{\theta}}{dz} \qquad \text{then} \qquad \frac{\partial}{\partial t}\frac{\partial^2 \theta'}{\partial x^2} = -\frac{d\bar{\theta}}{dz}\frac{\partial^2 w'}{\partial x^2}$$

(differentiated twice w/r/t *x*, rearranged)

Final steps...

$$\frac{\partial}{\partial t}\left[\frac{\partial^2 w'}{\partial x^2} - \frac{\partial^2 u'}{\partial z \partial x}\right] - \frac{g}{\bar{\theta}}\frac{\partial^2 \theta'}{\partial x^2} = 0$$

differentiate w/r/t *t* again and plug in expressions from last slide

$$\boxed{\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2}\right] + N^2\frac{\partial^2 w'}{\partial x^2} = 0}$$

$$\text{where } N^2 = \frac{g}{\bar{\theta}}\frac{d\bar{\theta}}{dz}$$

Pendulum equation.... note only *w* left

Solving the pendulum equation

- We expect to find waves -- so we go looking for them!
- Waves are characterized by period P, horizontal wavelength L_x and vertical wavelength L_z
- Relate period and wavelength to frequency and wavenumber

$$\omega = \frac{2\pi}{P} \quad k = \frac{2\pi}{L_x} \quad m = \frac{2\pi}{L_z}$$

A wave-like solution

$$w' = \hat{w} e^{i(kx+mz-\omega t)} \equiv E$$

where...

$$\hat{w} \quad \text{wave amplitude}$$

This is a combination of cosine and sine waves owing to Euler's relations

$$\begin{aligned} e^{iq} &= \cos q + i \sin q \\ e^{-iq} &= \cos q - i \sin q \end{aligned}$$

Differentiating the wave-like function

$$\begin{aligned}
w' &= \hat{w}e^{i(kx+mz-\omega t)} \\
\frac{\partial w'}{\partial x} &= \frac{\partial}{\partial x}\left[\hat{w}e^{i(kx+mz-\omega t)}\right] \\
&= \hat{w}e^{i(kx+mz-\omega t)}\frac{\partial}{\partial x}\left[i(kx+mz-\omega t)\right] \\
&= \hat{w}ike^{i(kx+mz-\omega t)}
\end{aligned}$$

now differentiate again

$$\begin{aligned}
\frac{\partial^2 w'}{\partial x^2} &= \frac{\partial}{\partial x}\left[\hat{w}e^{i(kx+mz-\omega t)}\right] \\
&= \hat{w}i^2k^2e^{i(kx+mz-\omega t)} \\
&= -\hat{w}k^2e^{i(kx+mz-\omega t)}
\end{aligned}$$

Differentiating the wave-like function

now differentiate twice with respect to time

$$\frac{\partial}{\partial t}\frac{\partial^2 w'}{\partial x^2} = \frac{\partial}{\partial t}\left[-\hat{w}k^2 e^{i(kx+mz-\omega t)}\right]$$

$$= -\hat{w}k^2 i\omega e^{i(kx+mz-\omega t)}$$

$$\frac{\partial^2}{\partial t^2}\frac{\partial^2 w'}{\partial x^2} = \underline{\hat{w}}k^2\omega^2 \underline{e^{i(kx+mz-\omega t)}}$$

Do same w/r/t z, and the pieces assemble into a very simple equation

$$\omega^2(k^2+m^2) - N^2k^2 = 0$$

The dispersion equation

$$\omega = \pm \frac{Nk}{\sqrt{(k^2 + m^2)}}.$$

A stable environment, disturbed by an oscillating parcel, possesses waves with frequency (period) depending the stability (N) and horizontal & vertical wavelengths (k, m)

Wave phase speed

$$c_x = \frac{\omega}{k} = \pm \frac{N}{\sqrt{(k^2 + m^2)}}$$

Example:

Wave horizontal wavelength 20 km vertical wavelength 10 km and stability N = 0.01/s

$$\begin{aligned} k = \frac{2\pi}{L_x} &= \frac{2\pi}{20000} = 3.14 \cdot 10^{-4} \\ m = \frac{2\pi}{L_z} &= \frac{2\pi}{10000} = 6.28 \cdot 10^{-4} \\ c_x &= \pm 14.24 \text{ m/s} \end{aligned}$$

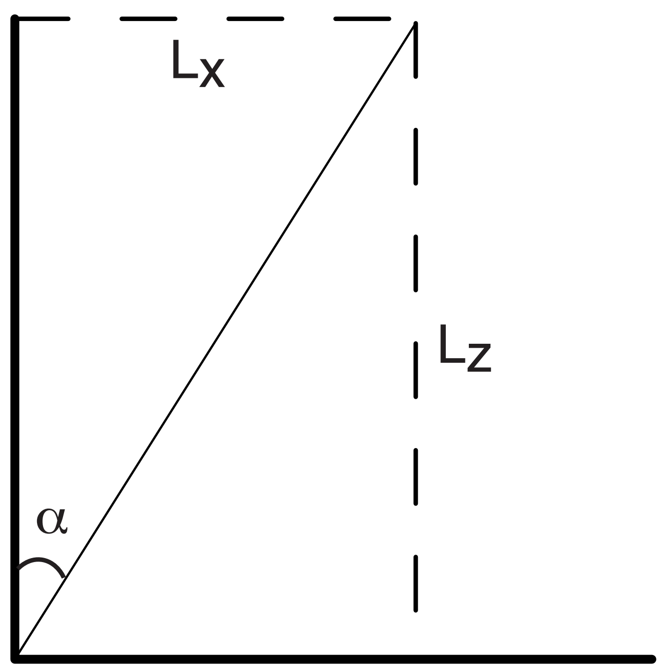
Wave phase speed

$$c_x = \frac{\omega}{k} = \pm \frac{N}{\sqrt{(k^2 + m^2)}}$$

Note as you make the environment more stable waves move *faster*.
Does that make sense?

Note that TWO oppositely propagating waves are produced.

Wave phase tilt



$$\begin{aligned}\alpha &= \arctan\left[\frac{L_x}{L_z}\right] \\ &= \arccos\left[\frac{L_z}{\sqrt{L_x^2 + L_z^2}}\right] \\ &= \arccos\left[\frac{k}{\sqrt{k^2 + m^2}}\right] \\ &= \arccos\left[\frac{\omega}{N}\right]\end{aligned}$$

Wave tilt with height depends on forcing frequency and stability (e.g., smaller ω smaller cos α larger tilt)