A&OS 101 Augmentation of Newton's laws for Earth's rotation

Fall, 2005 – Fovell

Newton's second law relates the acceleration of an object to the sum of the forces uper unit mass acting on the object; i.e.,

$$\vec{a} = \sum \frac{\vec{F}}{m}.$$

We recall that acceleration represents a change of speed and/or direction. Newton's law applies in an *inertial* reference frame. Such a reference frame can move, but it cannot rotate, have curvature or change speed as a function of time. That is, the reference frame cannot have accelerations associated with changes in speed and/or direction of the coordinate axes.

The Earth is not an inertial reference frame. Our coordinate system, which is most naturally affixed to the surface of the Earth, has accelerations owing to both rotation and sphericity. Rotation causes our coordinate axes to shift in time at a given location; sphericity causes them to be oriented differently at different places at the same time.

To apply Newton's law to our situation, we need to bolt on terms accounting for our two complicating factors, rotation and sphericity, which are illustrated on the left and right panels of Fig. 1, respectively. The result will be an equation we can use to understand motions in our favored reference frame. This handout specifically discusses the effect of coordinate system rotation on Newton's law. Sphericity is treated in a separate handout.

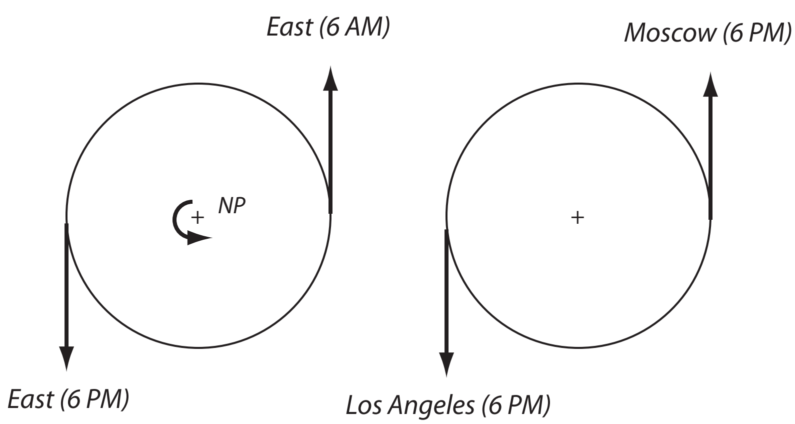

Figure 1: Orientation of the coordinate axis defining east as a function of time at single location (left) and as a function of longitude at a representative time (right).

Following, I describe the highlights of the transformation of Newton's law to a rotating reference frame. Let $\vec{A}$ be a generic vector, possibly representing velocity or position. Then, $\frac{d_a\vec{A}}{dt}$ represents the how $\vec{A}$ appears to change as seen from an inertial reference frame, such as outer space. The subscript "a" refers to this absolute reference frame. Meanwhile, $\frac{d\vec{A}}{dt}$ represents how $\vec{A}$ appears to change from our perspective here on the rotating Earth. This is our major interest.

A quickie example is provided in the left panel of the figure above. The space-based observer sees the coordinate axis we call east as shifting in time, and this change in direction implies the acceleration desisgnated as $\frac{d_a\vec{A}}{dt}$. However, from our perspective at a fixed location, what we see as the eastward direction does not change with time. Thus, in this example, $\frac{d\vec{A}}{dt} = 0$.

We relate these two accelerations as

$$\frac{d_a\vec{A}}{dt} = \frac{d\vec{A}}{dt} + \vec{\Omega}\times\vec{A}, \tag{1}$$

where $\vec{\Omega}$ is the Earth's rotation vector, constituting the spin axis perpendicular to the surface at the poles. The term on the left hand side (LHS) is the change in $\vec{A}$ seen from space, and Newton's laws apply to this term. The first term on the right hand side (RHS) is *what we want*, the change in $\vec{A}$ seen from Earth. The second RHS term is the rotation of the Earth as seen from space.

Examples: Say the Earth is not rotating (sphericity not being considered yet). Then $\vec{\Omega} = 0$ and $\frac{d_a\vec{A}}{dt} = \frac{d\vec{A}}{dt}$. The space observer and us see the same thing, and Newton's laws do not need to be augmented. Or, instead, let the Earth rotate and take $\vec{A}$ to be your position vector extending from the center of the Earth to your present location. Then $\frac{d\vec{A}}{dt}$ is your velocity relative to the rotating Earth. Further, let's say you are at rest, so $\frac{d\vec{A}}{dt} = 0$. The space observer sees you move because the Earth itself is rotating. Thus, in this situation,

$$\frac{d_a\vec{A}}{dt} = \vec{\Omega}\times\vec{A}; \tag{2}$$

the velocity seen from space is only the Earth's own spin velocity. If you are moving relative to the rotating Earth, then (1) applies. I will call this the "tool".

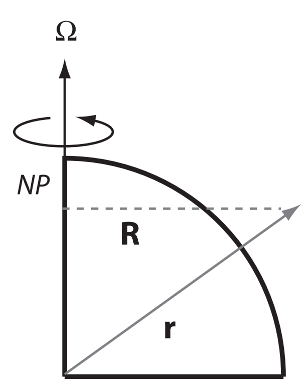

Figure 2: Vectors indicating position ($\vec{r}$) and distance from spin axis ($\vec{R}$) extending to the location shown.

A practical use of the tool is developed below. Again, let $\vec{A}$ be your position vector, but label it as $\vec{r}$, shown in Fig. 2. The vector $\vec{R}$ indicates your distance from the spin axis. In this case, (1) becomes

$$\frac{d_a\vec{r}}{dt} = \frac{d\vec{r}}{dt} + \vec{\Omega}\times\vec{r}. \tag{3}$$

We define $\vec{U}_a = \frac{d_a\vec{r}}{dt}$ as absolute velocity, seen from space. (It is the absolute derivative of your position, including the Earth's own rotational velocity.) Also, let $\vec{U} = \frac{d\vec{r}}{dt}$ be relative velocity…that is. velocity relative to the Earth's surface, Then (3) becomes

$$\vec{U}_a = \vec{U} + \vec{\Omega}\times\vec{r}. \tag{4}$$

Absolute velocity equals relative velocity plus Earth velocity.

Use the tool on (4), yielding

$$\frac{d_a\vec{U}_a}{dt} = \frac{d\vec{U}_a}{dt} + \vec{\Omega}\times\vec{U}_a. \tag{5}$$

and substitute in (4) on the RHS. After expanding terms, we obtain

$$\frac{d_a\vec{U}_a}{dt} = \frac{d\vec{U}}{dt} + \left[\frac{d\vec{\Omega}}{dt}\times\vec{r}\right] + \left[\vec{\Omega}\times\frac{d\vec{r}}{dt}\right] + (\vec{\Omega}\times\vec{U}) + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}). \tag{6}$$

Along the way, the chain rule was used to break up $\frac{d}{dt}(\vec{\Omega}\times\vec{r})$. Note that the Earth's spin velocity is constant (to a very, very good approximation), and thus $\frac{d\vec{\Omega}}{dt} = 0$. Further note that $\frac{d\vec{r}}{dt} = \vec{U}$. Therefore there are two instances of $\vec{\Omega}\times\vec{U}$ on the RHS…that's Coriolis (or, at least -1 times Coriolis)! Without dealing explicitly with centrifugal force and angular momentum conservation, Coriolis force has appeared in our equation, simply and directly owing to the rotation of our coordinate system. Finally, note that $\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) = -\Omega^2\vec{R}$, the centripetal force owing to Earth rotation (see separate handout). It's centripetal at this point because the vector is pointing in the $-\vec{R}$ direction, but this will change soon. Thus, we have

$$\frac{d_a\vec{U}_a}{dt} = \frac{d\vec{U}}{dt} + 2\vec{\Omega}\times\vec{U} - \Omega^2\vec{R}. \tag{7}$$

Newton's laws plug into the LHS term, and are augmented by accelerations relative to the rotating Earth, the negative of the Coriolis force, and the centripetal force.

We want $\frac{d\vec{U}}{dt}$. Thus, rearranging the equation, we basically find that the local acceleration in *our* favored reference frame is equal to Newton's law plus Coriolis plus centrifugal force. The $\Omega^2\vec{R}$ term is now called centrifugal force because it's pointing outward from the spin axis.

At this point, we need to explicitly specify the forces participating in Newton's law, which determines the change in absolute velocity as seen from space. The RHS terms in this equation

$$\frac{d_a\vec{U}_a}{dt} = \sum\frac{\vec{F}}{m} = -\frac{1}{\rho}\nabla p + \vec{g}^* + \vec{F}_r \tag{8}$$

represent pressure gradient force (PGF), true gravity and friction, respectively. Plugging these into (7) and solving for $\frac{d\vec{U}}{dt}$, our major interest, produces:

$$\frac{d\vec{U}}{dt} = -\frac{1}{\rho}\nabla p + 2\vec{\Omega} \times \vec{U} + \underbrace{\vec{g}^{*} + \Omega^{2}\vec{R}}_{\text{apparent gravity}} + \vec{F}_{r}. \tag{9}$$

The third and fourth terms on the RHS are combined to form $\vec{g}$, the apparent gravity.