AOS 103 - Physical Oceanography

Calculus Tools Refresher

Daniel Dauhajre
Julia Hazel
Andrew Stewart

Disclaimer: *AOS 103 is not necessarily a math-heavy course, but the mathematical concepts presented here will only help you in understanding the physical phenomena discussed in the class. Using these mathematical tools, you should be aiming to apply systems-level thinking to the concepts rather than memorization.*

Why do we need calculus (why do you need it for this class)?

Things in the ocean are always changing. Think of a local example in LA by imagining measuring water temperature from 2 buckets of water off 2 piers: Hermosa Beach and Malibu. If you did this measurement at the same time, there is a strong likelihood that the temperature of water in each bucket would be different. Similarly, the temperature at those 2 places will not be the same for 24 hours straight (the usual pattern is colder at night and warmer during the day). Simply put, the temperature changes in space and time. Calculus is the study of how things change, and so we use it to study oceanic processes...because everything in the ocean is always changing in space or time. Representing the evolution of the oceanic fields in a mathematical form (as **partial differential equations**) allows us to 1) prove or disprove the mechanisms that drive oceanic flows and 2) explain and/or predict behavior in the ocean.

Calculus Tools and Concepts

In this class you won't necessarily be directly deriving the formal equations that represent the way the ocean works. You will be required to understand and apply equations that are presented. One way to know if you really understand the math (and any concept really) is to be able to explain it to someone else in words from the ground up. You will get practice at this in discussion sections where you will have the chance to teach concepts to the class on the board; this can be one of the most effective ways to absorb the material...which will obviously result in doing well on the exams.

The following topics are the calculus tools and concepts that should give you the appropriate mathematical foundation for understanding the physical phenomena presented in this course:

- Differentiation and integration in one variable
- Partial derivatives
- Area and volume integrals (control volume)
- Gradient, Divergence, Curl (these will not explicitly come up as much in the class, but it will definitely help to understand what they mean and how they apply to oceanic fields)

The Math 3B/31B pre-req for the course should mean that everyone is familiar with these concepts, however it is possible that some may have forgotten or become unfamiliar with some of these items. Control volumes are probably something you may have never seen before but they are just a way to think about volume integrals in the ocean.

Differentiation and Integration in One Variable

Differentiation

A derivative is just the fancy Calculus way to calculate change over a given axis. To refresh your memory on the mechanics of derivatives, here is an example with a made up function that represents the horizontal temperature structure of an oceanic phenomenom known as a temperature filament (a patch of cold water bordered by patches of warm water):

$$T(x) = T_0 - \delta T_0 \mathrm{exp}(-\mathrm{x}/\mathrm{L})^2 \tag{1}$$

Where $T_0, \delta T_0, L$ are constants that have been chosen to make the values of the function (Figure 1) similar to real ocean values. The derivative of $T(x)$ with respect to x is as follows:

$$\frac{dT}{dx} = \frac{2x\delta T_0}{L}\mathrm{exp}(-\mathrm{x}/\mathrm{L})^2 \tag{2}$$

If you are unsure of how that derivative was calculated revisit exponential derivative rules

The most important aspect of derivatives in this course will have to do more with visualization and interpretation of derivatives as opposed the mechanics of taking a derivative. The figure below shows the plot of the function above and its derivative:

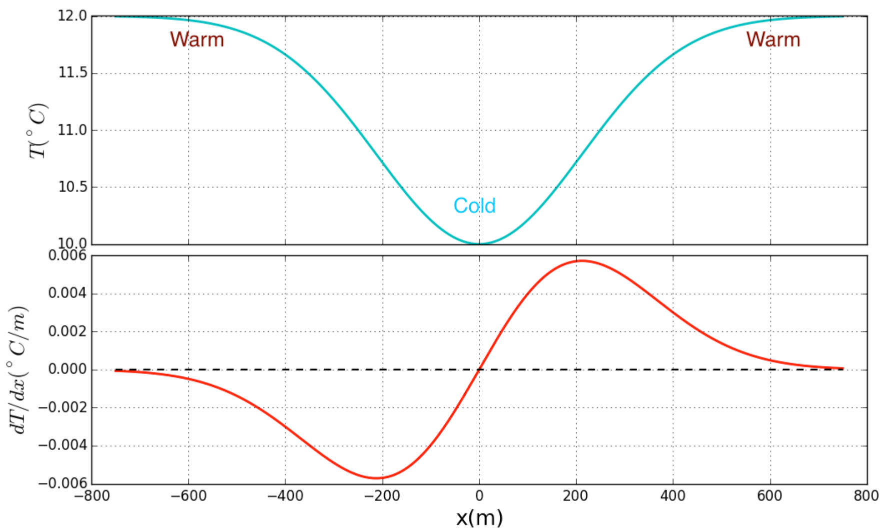

Figure 1: Plot of temperature (top) and derivative of temperature (bottom).

Visually, the derivative (dT/dx) follows the rise and fall of temperature along the x-axis. On the left side of the origin and moving from left to right (along the x-axis), temperature **changes** from warm to cold ($dT/dx < 0$) and on the right side temperature **changes** from cold to warm ($dT/dx > 0$). The derivative dT/dx just says **how fast or slow the temperature is changing**.

If you have trouble putting the equations into this physical perspective, sometimes it can help to literally translate what the equation is telling you.

In this simple case: dT = temperature change, and dx = change in distance. So dT/dx = rate of change of temperature along a distance. Large values of dT/dx mean temperature is changing fast and vice versa.

Something you may be asked to do on homework is to calculate the derivative from *discrete* (point-wise) measurements. This actually doesn't involve any formal calculus, all you have to do is subtract two values at two different points. We can visualize this with our same function:

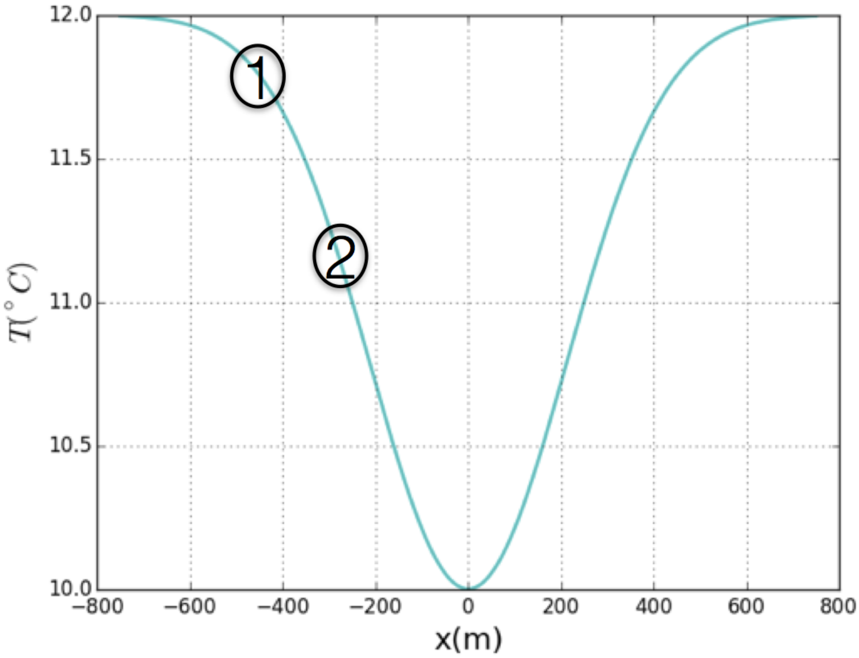

Figure 2: Illustration to show how to evaluate derivatives *discretely* (at individual points, here labeled 1 and 2).

The discrete calculation of the derivative would be as follows:

$$\frac{dT}{dx} = \frac{\Delta T}{\Delta x} = \frac{T_1 - T_2}{x_1 - x_2} \tag{3}$$

Integration

1D integration can be simply described as taking a sum over a certain range. Everyone is probably familiar with the description of **1D integration as the calculation of the area under a curve**; let's apply that to an oceanographic variable and give a physical interpretation.

Say that we wanted to calculate the amount of biomass across a certain distance in the ocean. Again, let's use a local example and imagine that we have measurements on a line that sits just offshore from Marina Del Rey(MDR) to Palos Verdes (PV) which are roughly $10km$ apart. This imaginary line (what we commonly call a *transect*) is shown in the figure below on the left panel (orange line).

Now imagine that we have a biomass measurements along this line ($B(x)$) in units of (kg/km)) at a depth of 20m below the surface. A very plausible way to take this measurement would be to drive a boat along that line with an instrument sitting at 20m depth measuring biomass. Let us now say that the biomass ($B(x)$) along that line happens to b given by the (again made up) function (which is plotted on the right panel of the figure below):

$$B(x) = 2 + 0.1x^5 + \cos(x) \tag{4}$$

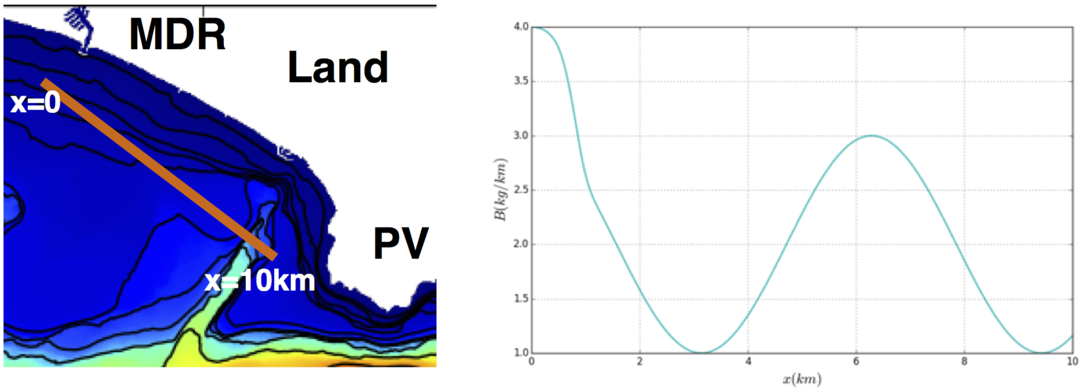

Figure 3: : Left: Illustration of transect extending from Marina Del Rey (MDR) to Palos Verdes (PV). Right: Plot of $B(x)$. For the map on the left colors and thin black lines are contour showing the bathymetry of the seafloor (i.e., the ocean depth)

We can calculate the total amount of biomass across a certain range [a,b] along the transect by integration:

$$\int_a^b B(x)dx = \int_a^b 2 + 0.1x^5 + \cos(x)dx = 2x + \frac{0.1x^6}{6} + \sin(x)\Big|_a^b = B_{tot} = \text{total biomass(kg)} \tag{5}$$

Note that this calculation is only telling us the biomass at a certain depth (which would have units of kg). If we wanted to know the biomass along the line covering all depths (not just at 20m), we would redo our calculation using an area integral, which will be discussed later.

Partial Derivatives

Partial derivatives are used for **fields that are multidimensional**. If you were to look at a snapshot of Santa Monica and San Pedro Bay from the sky, you would be looking at it in 2-dimensions. A snapshot of sea surface temperature (SST) from a regional ocean model simulation of this area is shown in the figure below:

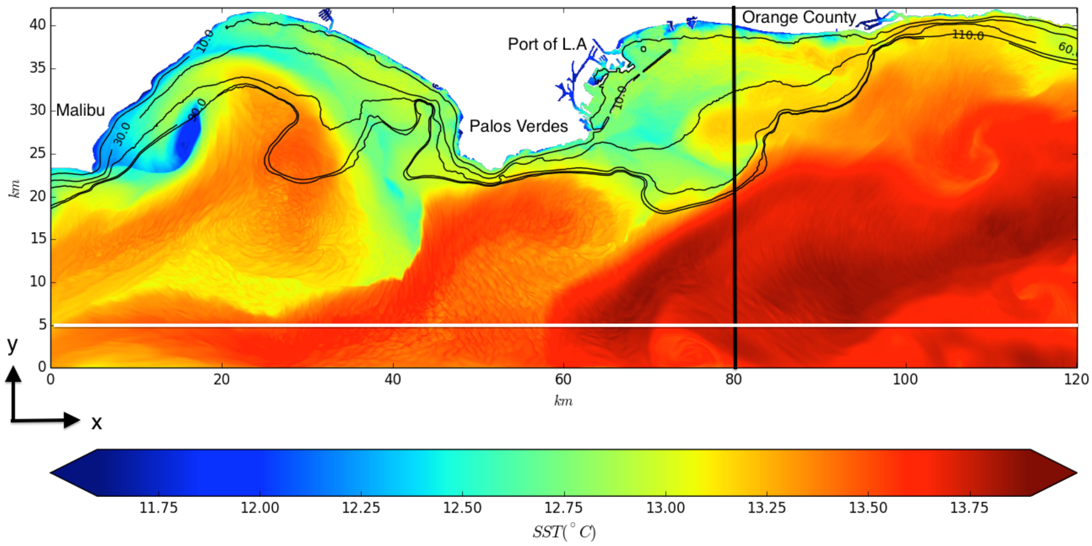

Figure 4: Snapshot of sea surface temperature (SST) of Santa Monica and San Pedro Bay in Southerin California (data taken from Regional Oceanic Modeling System (ROMS) simulation). Colors indicate temperature (blues = cool, reds = warm), with black contours indicated depth (white is land). The black and white lines across the domain indicate lines to visualize partial derivatives across. The $x$and y axes orientation are denoted in the bottom left-hand corner.)

The temperature in the snapshot above is **variable** in 2-dimensions in the horizontal (temperature of course varies with depth as well, but that dimension is not shown in the figure). That is, the colors change along both axes. Again, a very important skill to develop for this class is the visual interpretation of the variation of fields along dimensions. Let us denote the horizontal axis as the $x-$axis and the vertical axis as the $y-$axis.

If you were to draw a line along the vertical axis at the 80km mark on the x-axis (black line in figure), you would say that the temperature is generally decreasing from the offshore to nearshore (red colors to orange/green-blue). This of course is not an exact description as there is an oscillation of temperature in the nearshore (warm-cold-warm = orange-green-orange), but generally we can say, mathematically, that $\partial T/\partial y < 0$. (We denote the *partial* part of the partial derivative as ∂T instead of dT, which would denote a full derivative).

If you were to similarly draw a line parallel to the $x-$axis along the 5km mark on the $y-$axis (white line), you could say that temperature generally increases in that direction (orange/red to darker red from Malibu to Orange County). Again, mathematically we would state this as $\partial T/\partial x > 0$.

These statements are a description of the partial derivatives of temperature (*roughly speaking, i.e, not exact) in this snapshot. Partial derivatives express the rates of change of multidimensional functions along their given axes (dimensions).

Just like before, let us imagine that the temperature field is described by a made up function, but this time it is variable in both x and y:

$$T(x,y) = T = (\cos(2x+5))^2 + \sin(3y+2)$$

The partial derivatives that result from differentiation in x and y are as follows:

$$\frac{\partial T}{\partial x} = -4\cos(2x+5)\sin(2x+5) \quad (7)$$

$$\frac{\partial T}{\partial y} = 3\cos(3y+2) \quad (8)$$

(7) and (8) are derived by regular 1D differentiation with x treated as a variable and y treated as a constant for (7) and vice versa for (8). The meaning of each is simply **the temperature change along each respective axis**.

Area Integrals and Control Volumes

The snapshot below is from a computer model (made here at UCLA) of the Gulf stream. If you're unsure on what the Gulf Stream is, its just that "stream" of water flowing northeast along the red contours (that indicate warm temperature) in the snapshot.

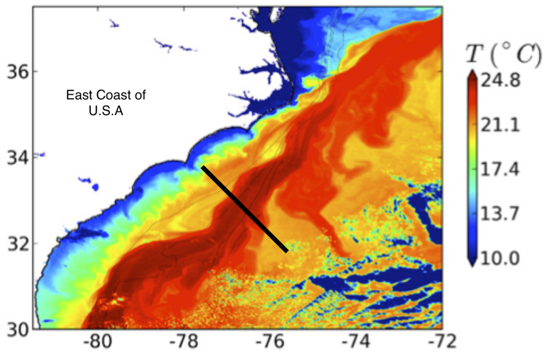

Figure 5: Sea surface temperature (colors) from a ROMS simulation of the Gulf Stream. The numbers on the bottom axis denote longitude and the numbers on the vertical axis denote latitude. The black line denotes the horizontal width of the area where we wish to calculate transport through.

Let's suppose that we wanted to know how much of some ocean property (water, mass, salt, nutrients, oil!) is transported across some part of the Gulf Stream, a very fast and globally important current. And let's say that we want to know the transport that happens across some horizontal distance (denoted by the black line in the figure) and down to a depth of 1km. **We can do this calculation using an area integral.**

Area Integrals

We can set up and visualize our transport calculation in Figure 6. We are essentially dropping a rectangular 'frame' (what we call a 'cross-section') into the water from above the ocean and letting it sit in the (x, z) plane (i.e one axis in the horizontal, one in the vertical going down to depth).

Transport is achieved by advection (bulk movement due to fluid flow). Thus, all we need to calculate transport are the following:

- **Area of the domain** to calculate transport through (width times height of our cross-section) $= dx \times dz$
- **Property transported** (let's keep it simple and let it be water; but we could just as easily calculate transport of heat, salt, nutrients, etc.)
- **Velocity normal to the face of the cross-section** (i.e flowing through the cross-section) $= v$

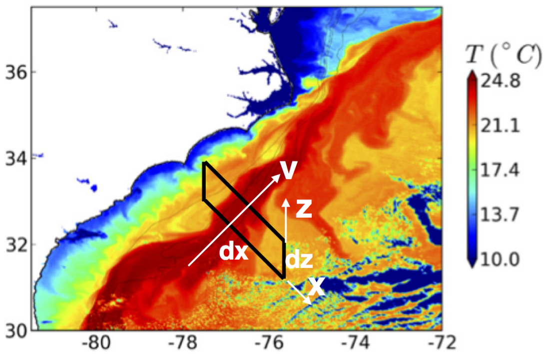

Figure 6: Illustration of set up for calculating transport. The 2D domain has a vertical height dz and horizontal width dx, with a velocity v transporting (advecting) material through it.

Since we have decided to calculate the amount of water transported, we are calculating the **volume transport**, which will give us units of (m^3/s). Volume transport has the following, very simple, equation:

$$volume\ transport = velocity\ \times\ area\ \ ===>\ \frac{m}{s} \times m^2 = \frac{m^3}{s} \tag{9}$$

If the velocity throughout the cross-section is a continuous function in (x, z) (e.g. $v(x, z) = x^2 + e^{-z}$) we can formally write this as an area integral:

$$total\ volume\ transport = \int\int v(x, z)dxdz \tag{10}$$

The double integral is used because we are summing over 2 dimensions: x and z. We can easily extend this formulation for mass transport by adding in a density profile, $\rho(x, z)$:

$$total\ mass\ transport = \sum_{i=1}^{N} v_i \rho_i dx_i dz_i = \int\int v(x, z)\rho(x, z)dxdz \tag{11}$$

Which will give us units of (kg/s). *Density has units of* $\frac{kg}{m^3}$. We can do analogous formulations for heat, nutrient concentration, salt and other ocean properties (all we would do is put in the appropriate factor in the integral and get different transport units).

*****NOTE: you will do this sort of calculation quite a bit in this class (probably even on an exam...probably not even in the scary double integral form, but the simple form of Eq (9)). So it is important you understand it. By that I mean, it will probably help if you really try to physically comprehend what is going on.**

Control Volume

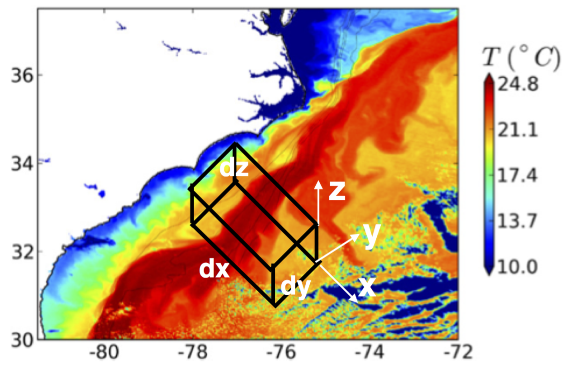

Figure 7: Illustration of set up for control volume.

We can add complexity to our cross-section setup in Figure 6 by creating a 3D box that has a length dy (shown above in Figure 7). In fluid mechanics, this imaginary box is often called a **control volume** and it can be rigid or deformable depending on the assumptions you make for your system. With our area integral method for calculating transport, we could get the transport through each face of the box. However, this would mean that we will have to take into account velocities flowing normal to (through each) face (not just the $v(x, z)$ from before).

Once we have the transport though each face, we could add them up. If that sum is not 0, that would mean the water flowing in does not equal the water flowing out of the control volume. It actually turns out that, we frequently assume that the sum must be 0. We say that the fluid in the control volume (and more importantly in the ocean) is **conserved** (in other words, there are no holes in the box). To fully understand this concept we need to talk about 3D fluid velocities and their properties...

Gradient, Divergence, Curl

Before we begin our discussion of 3D fluid flows, we need to establish the difference between **scalars** and **vectors**.

- **Scalars** are fully described by a numerical value **(magnitude)**

- Ocean scalars include temperature, salinity, density, nutrient concentration

- **Vectors** are fully described by BOTH a **magnitude and direction**
  - Ocean vectors include velocity and gradients of the above scalar fields

We write the ocean's 3D velocity (**U**) as a vector:

$$\mathbf{U} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} \tag{12}$$

With the following coordinate conventions in space:

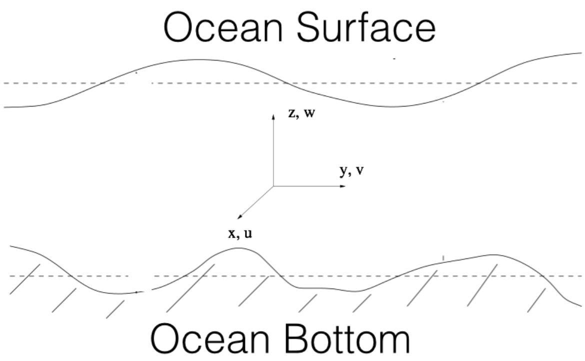

Figure 8: Illustration of the coordinate system corresponding to the ocean 3D velocity. The horizontal velocities (u,v) flow through the page and to the right/left on the page, respectively. Vertical velocity (w) flows up or down on the page.

<u>Gradient</u>

If a field (say, temperature again) is a 3D function, the gradient is calculated by simply taking the partial derivative along each dimension and representing the partial derivatives in vector form:

$$T(x, y, z) = 2x + \sin(y) + e^{-z} \tag{13}$$

We use ∇ (or grad) to denote the gradient operator:

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \tag{14}$$

$$\mathrm{gradT} = \nabla \mathrm{T} = \frac{\partial \mathrm{T}}{\partial \mathrm{x}}\mathbf{i} + \frac{\partial \mathrm{T}}{\partial \mathrm{y}}\mathbf{j} + \frac{\partial \mathrm{T}}{\partial \mathrm{z}}\mathbf{k} = 2\mathbf{i} + \cos(\mathrm{y})\mathbf{j} - \mathrm{e}^{-\mathrm{z}}\mathbf{k} \tag{15}$$

You can think about what ∇T means geographically as follows: ∇T **will be a vector that is aligned with the direction along which T changes most rapidly**. You can see an example of this in Figure 9 (below), in which ∇T is aligned across the regions where temperature (colors) change most rapidly (the snapshot below shows an example of a 'temperature front'). The convention used is to point the vector in the direction in which temperature is rising most quickly (at that point). So, for the point denoted by the 'x', the most rapid temperature cold to warm (orange to purple) temperature change is in the onshore to offshore direction.

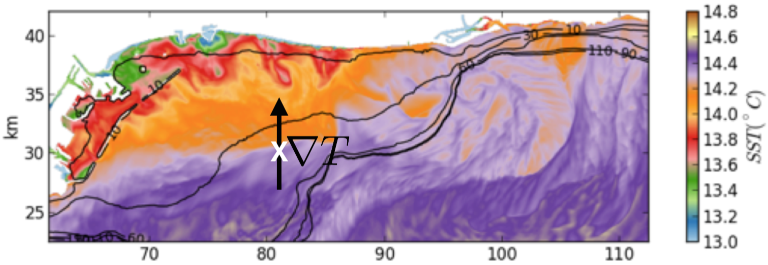

Figure 9: Illustration of temperature gradient vector at a point X. The gradient vector points in the direction in which temperature changes most quickly

When we want **the spatial variation of a scalar we use the gradient** operator. If we want to look at the full spatial variation of a vector field, we need to use the gradient operator combined with vector operations ($\nabla\cdot$ and $\nabla\times$). *This is not to say that we cannot take the gradient of, say, the x-component of velocity (u), which by itself can be thought of as a scalar...but curl ($\nabla\times$) and divergence ($\nabla\cdot$) of vector fields are typically more meaningful.*

Divergence

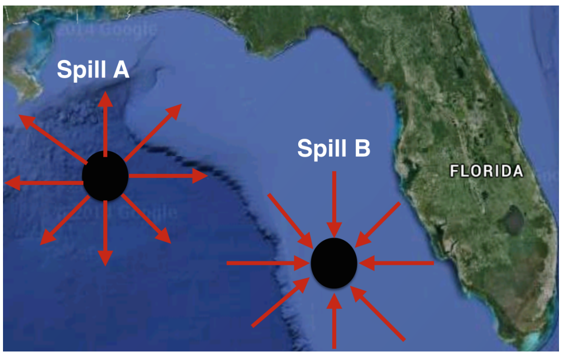

Figure 10: Oil patches with separate velocity fields surrounding each, denoted by the red arrows.

Imagine that there is are 2 oil spills at separate locations in the surface of the ocean, and for simplicity, imagine that this oil on the surface of the ocean (oil is less dense than ocean water, therefore it will float at the surface) has an initial shape of a perfect circle and it is not diffusing through the water. That is, the only way the oil will move is through advection by the ocean's velocity field. Let's further simplify things and say that the oil's viscosity has no effect on the ocean's flow. Also, because oil is substantially less dense than ocean water, we can safely assume that the only velocity advecting the oil is the horizontal (x, y) velocity (vertical velocity will not transport the oil). This scenario is crudely depicted in Figure 10. The final assumption we will make (again for the sake of the example) is that the velocity fields surrounding each patch of oil (red arrows) are independent of each other; that is, the arrows on Spill A do not change the arrows on Spill B, and vice versa.

We can do a simple thought experiment to predict what will happen to each patch of oil given each velocity field. Again, remember that the only way the oil moves is through advection by velocity. For Spill A, given the outward pointing arrows, we can easily imagine that the oil will spread symmetrically outward from its initial point source (circle). Conversely for Spill B, we can imagine that the oil will remain in place (and the circle will possibly shrink in diameter) due to the velocity pushing in on the patch. We can say that the velocity field surrounding Spill A is **divergent** and the velocity field surrounding Spill B is **convergent**.

All we are doing when we calculate the divergence of a vector field (in most cases fluid velocity), is mathematically stating whether more fluid is flowing away from that point than towards it.

The divergence operator ($\nabla\cdot$ or div) can be formally defined as follows:

Given a vector field:

$$\mathbf{U} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} \tag{16}$$

The **divergence is defined as the dot product of the gradient operator with the vector field**:

$$\nabla \cdot \mathbf{U} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle u, v, w \rangle = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{17}$$

Let's say that we have a horizontal 2D velocity field defined by:

$$\mathbf{U}_{2D} = x^2y\mathbf{i} + y^2x\mathbf{j} \tag{18}$$

The divergence is calculated as follows:

$$\mathrm{div}\mathbf{U_{2D}} = \nabla \cdot \mathbf{U}_{\mathrm{2D}} = \frac{\partial}{\partial \mathrm{x}}(\mathrm{x}^2\mathrm{y}) + \frac{\partial}{\partial \mathrm{y}}(\mathrm{y}^2\mathrm{x}) = 2\mathrm{xy} + 2\mathrm{xy} = 4\mathrm{xy} \tag{19}$$

Note that the **divergence is a scalar**, not a vector.

Now let's add in a vertical velocity and take the divergence of a fully 3D fluid velocity field:

$$\mathbf{U}_{3D} = x^2y\mathbf{i} + y^2x\mathbf{j} - 4xyz\mathbf{k} \tag{20}$$

$$\nabla \cdot \mathbf{U}_{3D} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(y^2x) + \frac{\partial}{\partial z}(4xyz) = 2xy + 2xy - 4xy = 0 \tag{21}$$

The fact that the divergence of the 3D fluid velocity equals 0 is not by mistake, but is done to bring up an important assumption made in physical oceanography. A very general and important assumption in physical oceanography is that we assume **the fluid in the ocean is incompressible**. Mathematically this assumption results in the divergence of the 3D velocity summing to zero:

$$\nabla \cdot \mathbf{U} = 0 \tag{22}$$

We will probably talk about this concept later on in the course.

Curl

In the ocean, currents often exhibit rotational tendencies. This is visualized in the figure below, which shows a vortex off the coast of Malibu. The red colors represent a quantity called **relative vorticity**. Vorticity is a very important quantity in physical oceanography as it can be used to easily visualize flow fields and it is used a great deal in the formulation of theories for fluid flow in the ocean. Vorticity is calculated by taking a component of the curl of the velocity field.

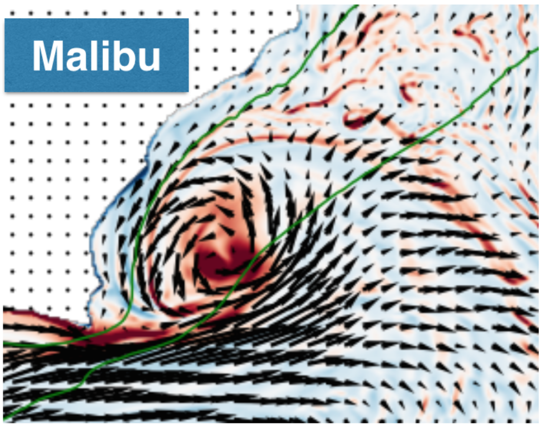

Figure 11: Surface horizontal velocity (arrows) and surface relative vorticity (colors) off the coast of Malibu. The velocity field is from the output of ROMS simulation.

Simply put, **the curl is a calculation of the amount of rotation in a flow(vector) field**. Unlike the divergence, the result of a the curl operator is a vector that describes rotation in each plane that there is fluid flow. The curl operator ($\nabla\times$ or curl) for a 3D flow field is defined as follows:

Given the same 3D velocity field from before:

$$\mathbf{U} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} \tag{23}$$

The **curl is the cross product of the gradient operator with the vector field**:

$$\text{curl}\mathbf{U} = \nabla \times \mathbf{U} = \left\langle \frac{\partial}{\partial \text{x}}, \frac{\partial}{\partial \text{y}}, \frac{\partial}{\partial \text{z}} \right\rangle \times \langle \text{u}, \text{v}, \text{w} \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \tag{24}$$

$$\nabla \times \mathbf{U} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \mathbf{k} \tag{25}$$

Again **note that $\nabla \times \mathbf{U}$ is a vector**, not a scalar.

The surface vorticity shown in Figure 8 is just the last component $\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k}$ of the curl operator (it is also normalized by the rotation of the Earth, but it is the curl operation that gives the spatial structure of the colors in the figure). Vorticity is an indicator of rotation in the

(x, y) plane *about (around)* the vertical (z) axis; it is also a great indicator of **velocity shear**. Shear in velocity fields can be caused by velocity vectors opposing each other, or, more generally, gradients in the strength and/or direction of the velocity itself. Two examples of shear flows are shown below:

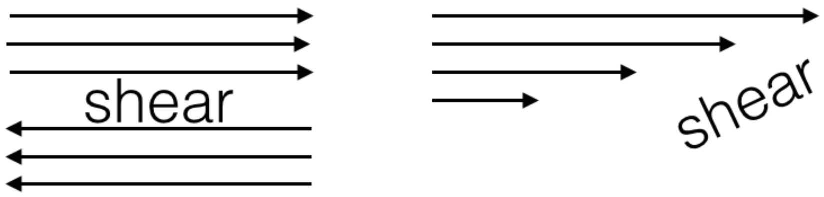

Figure 12: Flow profiles that can lead to velocity shear. The flow profile on the left has equal velocity magnitude with opposing directions. The flow profile on the right has velocity with uniform direction, but a top to bottom drop off in magnitude.

Calculus Tools - Short Version

Look back to sections in long version for example figures and plots. Visual interpretation of data and how these mathematical concepts apply to them is very important for this class.

Why do you need calculus in this class?

- Things in the ocean are always changing
- Calculus provides tools to calculate how things change.

Differentiation (Derivatives)

- Derivatives calculate the rate of change of a variable along an axis
- Point-wise definition: $\frac{dT}{dX} = \frac{\Delta T}{\Delta x} = \frac{T_2 - T_1}{x_2 - x_1}$
- Example: $T(x) = 3x^4 + e^{-2x} + \sin(3x+1)$, $\frac{dT}{dx} = 12x^3 - 2e^{-2x} + 3\cos(3x+1)$

Integration

- Integrals sum a variable over a given axis (opposite of a derivative)
- Example: $B(x) = 2x^5 + \cos(x)$
- $\int_a^b B(x)dx = \int_a^b 2x^5 + \cos(x)dx = \frac{x^6}{3} + \sin(x)\Big|_a^b$

Partial Derivatives

- Partial derivatives are used to calculate derivative of **multidimensional** variables
- Example: $T(x,y) = T = (\cos(2x+5))^2 + \sin(3y+2)$
- $\frac{\partial T}{\partial x} = -4\cos(2x+5)\sin(2x+5)$
- $\frac{\partial T}{\partial y} = 3\cos(3y+2)$

Area Integrals and Control Volume

- Area (double) integrals can be used to calculate transport of a property through a 2D area
- Given a 2D area (cross-section) in the ocean (Figure 6) and the velocity through that area we can calculate volume transport
- $volume\ transport = velocity\ \times\ area\ ===>\ \frac{m}{s} \times m^2 = \frac{m^3}{s}$
- $volume\ transport = \int\int v(x,z)dxdz$
- Control volumes are the 3D extension of 2D cross-sections, and can be used to calculate transport via conservation laws

Scalars and Vectors

- **Scalars** are fully described by a numerical value **(magnitude)**
  - Ocean scalars include temperature, salinity, density, nutrient concentration
- **Vectors** are fully described by BOTH a **magnitude and direction**
  - Ocean vectors include velocity and gradients of the above scalar fields
- The 3D ocean velocity vector can be written as follows: $\mathbf{U} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$
- This vector represents flow along 2 horizontal (**i**, **j**) dimensions and 1 vertical (**k**) dimension (Figure 8)

Gradient

- The gradient is a vector that describes the rate of change of a variable along its dimensions (Figure 9)
- The symbol for the gradient operator is ∇
- $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$
- Example: $T(x, y, z) = 2x + \sin(y) + e^{-z}$
- $\nabla T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k} = 2\mathbf{i} + cos(y)\mathbf{j} - e^{-z}\mathbf{k}$

Divergence

- For a fluid (say water), calculating the divergence can tell us whether fluid is moving towards (converging) or away (diverging) from a point (Figure 10)
- The symbol for the divergence operator is $\nabla\cdot$
- The divergence operator is **applied to vectors and results in a scalar**
- $\nabla \cdot \mathbf{U} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle u, v, w \rangle = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$
- Example: $\mathbf{U}_{2D} = x^2y\mathbf{i} + y^2x\mathbf{j}$
- $\nabla \cdot \mathbf{U}_{2D} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(y^2x) = 2xy + 2xy = 4xy$

Curl

- Curl describes the amount of rotation in a field(for our purposes this field is fluid velocity...look at Figure 11)
- The symbol for the curl operator is $\nabla\times$
- The curl operator is **applied to a vector and results in a vector**
- $\mathbf{U} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$
- $\nabla \times \mathbf{U} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle u, v, w \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$
- $\nabla \times \mathbf{U} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k}$