Numerically Accurate treatment of bottom drag in ocean models with mode and time splitting

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- While representation of bottom drag term is essential for coastal modeling, the basic mathematical dilemma of handling an implicit no-slip bottom boundary condition in a mode-split model (whether split-explicit or implicit free-surface) is not satisfactorily solved within the oceanic modeling community, and, moreover, most current oceanic codes do not even allow this without a major algorithmic redesign.

- The essence of the problem is the splitting of two stiff operators – one is associated with the Barotropic Mode splitting, the other is due to the implicit no-slip boundary condition at the bottom – a situation reminiscent to the classical dilemma in incompressible flows, e.g., Dukowicz & Dvinsky, 1992.

- Vertical grid refinement toward the bottom – a standard modeling practice motivated by the need to resolve turbulent bottom boundary layer – exacerbates the splitting dilemma resulting in poor convergence.

- This presentation overviews the current modeling practices and proposes self-consistent algorithms to address both the errors due to time splitting and handling of the discrete of no-slip bottom boundary condition in the turbulent case where regularization length (viscous sublayer) is only marginally resolved or not resolved at all.

Bottom drag: Physics and Discretization Issues

model needs $$\Delta z_1 \cdot \frac{u_1^{n+1} - u_1^n}{\Delta t} = A_{3/2} \cdot \frac{u_2^{n+1} - u_1^{n+1}}{\Delta z_{3/2}} - r_D \cdot u_1^{n?} \qquad r_D = ?$$

where $u_1 \equiv u_{k=1}$ is understood in finite-volume sense $u_1 = \frac{1}{\Delta z_1} \int_{\text{bottom}}^{\text{bottom}+\Delta z_1} u\left(z'\right)\, \mathrm{d}z'$

from physics $\text{STRESS} = F(u), \qquad F = ?$

duality of u_*: it controls **both** bottom stress and vertical viscosity profile

$$\text{STRESS} = u_*^2\,, \qquad \text{and} \qquad A = A(z) = \kappa u_* \cdot (z_0 + z) \qquad z \to 0$$

roughness length z_0 = statistically averaged scale of unresolved of topography

constant-stress boundary layer $A(z) \cdot \partial_z u = \text{STRESS} = const = u_*^2$

$$\kappa u_* \left(z_0 + z\right) \partial_z u = u_*^2 \qquad \text{hence} \qquad u(z) = \frac{u_*}{\kappa} \ln\left(1 + \frac{z}{z_0}\right)$$

$$u_1 = \frac{u_*}{\kappa}\left[\left(\frac{z_0}{\Delta z_1} + 1\right) \ln\left(1 + \frac{\Delta z_1}{z_0}\right) - 1\right] \qquad \text{hence} \qquad u_* = \kappa \cdot u_1 / [...]$$

$$-r_D \cdot u_1 = -\kappa^2 \left|u_1\right| \cdot \left[\left(\frac{z_0}{\Delta z_1} + 1\right) \ln\left(1 + \frac{\Delta z_1}{z_0}\right) - 1\right]^{-2} \cdot u_1$$

$$r_D = \kappa^2 \,|u_1| \Bigg/ \left[\left(\frac{z_0}{\Delta z_1}+1\right)\ln\left(1+\frac{\Delta z_1}{z_0}\right)-1\right]^2$$

well-resolved asymptotic limit for $\Delta z_1/z_0 \ll 1$ is $\quad r_D \sim 4\kappa^2\,|u_1|\cdot\dfrac{z_0^2}{\Delta z_1^2}$

however in this case $\quad u(z) = \dfrac{u_*}{\kappa}\ln\left(1+\dfrac{z}{z_0}\right) \sim \dfrac{u_*}{\kappa}\cdot\dfrac{z}{z_0} \quad$ hence $\quad u_1 = \dfrac{u_*}{\kappa}\cdot\dfrac{\Delta z_1}{2z_0}$

resulting $\quad r_D \sim \kappa^2 u_* \cdot \dfrac{2z_0}{\Delta z_1} = \dfrac{A_{\text{bottom}}}{\Delta z_1/2} \quad$ in line with no-slip with laminar viscosity

unresolved $\Delta z_1/z_0 \gg 1$ limit $\quad r_D \sim \kappa^2\,|u_1| \Big/ \ln^2\left(\dfrac{\Delta z_1}{z_0}\right) \quad$ known as "log-layer"

- overall there is nothing unexpected
- smooth transition between resolved and unresolved
- avoids introduction of *ad hoc* "reference height" z_a, e.g., Soulsby (1995) formula $\mathrm{STRESS} = [\kappa/\ln(\mathrm{z_a}/\mathrm{z_0})]^2\cdot \mathrm{u}^2\big|_{\mathrm{z=z_a}}$ where $u\big|_{z=z_a}$ is hard (or impossible) to estimate from discrete variables
- in practice this differs by a factor of 2 from published formulas, e.g., Blaas (2007), with $z_a = \Delta z_1/2$, due to finite-volume vs. finite-difference interpretation of discrete model variables
- near-bottom vertical grid-box height Δz_1 is an inherent control parameter of r_D, making it impossible to specify "physical" quadratic drag coefficient, $r_D = C_{\mathrm{D}}\cdot|u|$

How large is $\dfrac{\Delta t \cdot r_D}{\Delta z_1}$?

$$\frac{\Delta t \cdot r_D}{\Delta z_1} = \underbrace{\frac{\Delta t \cdot |u_1|}{\Delta x}}_{\substack{\text{advective} \\ \text{Courant number}}} \cdot \kappa^2 \cdot \underbrace{\frac{\Delta x}{\Delta z_1} \Bigg/ \left[\left(\frac{z_0}{\Delta z_1} + 1\right) \ln\left(1 + \frac{\Delta z_1}{z_0}\right) - 1\right]^2}_{\text{purely geometric criterion}}$$

in unresolved case $\quad \dfrac{\Delta x}{\Delta z_1} \cdot \left[\kappa \Big/ \ln\left(\dfrac{\Delta z_1}{z_0}\right)\right]^2$

Typical high-resolution ROMS practice $h_{\text{min}} \sim 25m$, $N = 30...50$, hence $\Delta z \sim 1m$, $\Delta x = 1km$, and $z_0 = 0.01m$, $\kappa = 0.4$ estimates the above as 7.5.

- $\sim 50...100$ **in Bering Sea in our** $\Delta x = 12.5$**km Pacific simulation, even more in a coarser 1/5-degree**

It is mitigated by the bottom-most velocity Courant number ~ 0.1 but, still exceeds the limit of what explicit treatment can handle

- sigma-models are the most affected, but they are the ones which are mostly used when bottom drag matters
- vertical grid refinement toward the bottom makes this condition stiffer

Implicit treatment of $-\Delta t \cdot r_D \cdot u_1^{n+1}$ term: include it into implicit solver for vertical viscosity terms, however this interferes with Barotropic Mode (BM) splitting:

- Bottom drag can be computed only from full 3D velocity, but not from the vertically averaged velocities alone.
- Barotropic Mode must know the bottom drag term **in advance** as a part of 3D→2D forcing for consistency of splitting. This places computing vertical viscosity before BM, however, later when BM corrects the vertical mean of 3D velocities, it *destroys* the consistency of (no-slip like) bottom boundary condition.
- If BM receives bottom drag based on the most recent state of 3D velocity **before** BM, but the implicit vertical viscosity terms along with (the final) bottom drag are computed **after** BM is complete (hence accurately respecting the bottom boundary condition), this changes the state of vertical integrals of 3D velocities, interfering with BM in keeping the vertically integrated velocities in nearly non-divergent state.
- Current ROMS practice is to split bottom drag term from the rest of vertical viscosity computation. This limits the time step (or r_D itself) by the explicit stability constraint.

Ekman layer in shallow water:

$h = 10m$, $f = 10^{-4}$, $A_v = 2 \times 10^{-3} m^2/s$
$u_* = 6 \times 10^{-2} m/s$ ($\approx 5m/s$ wind),
non-slip at $z = -h$, $N = 30$

Top: Explicit, CFL-limited, bottom drag **before** Barotropic Mode (BM) for **both** r.h.s. 3D and for BM forcing ($\Rightarrow$ no splitting error); implicit step for vertical viscosity **after** with bottom drag excluded ($\Rightarrow$ undisturbed coupling of 2D and 3D);
need $r_D < \Delta z_{\text{bottom}} / \Delta t_{3D}$ for stability

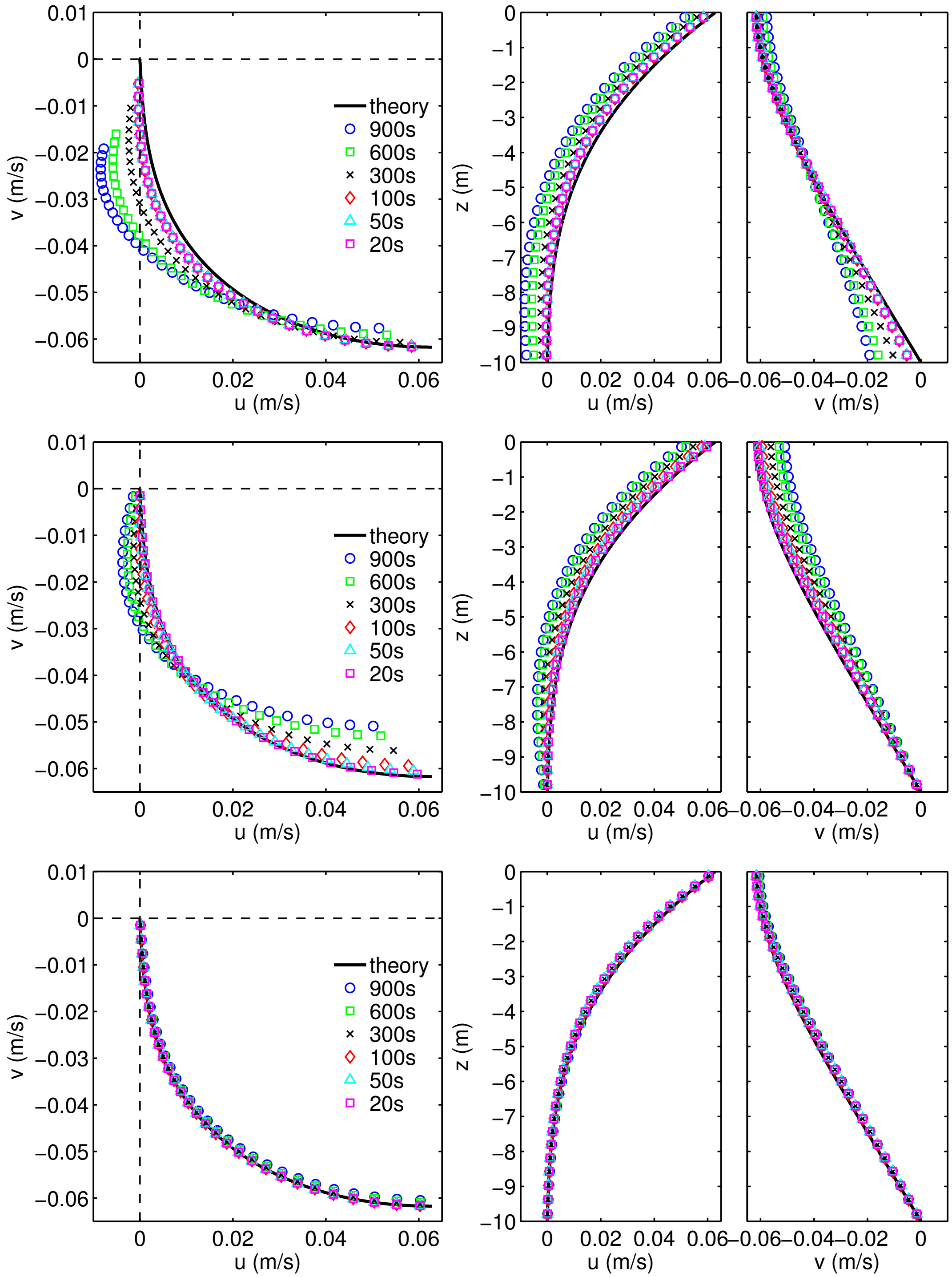
Middle: Unlimited drag **before** BM applies for BM forcing **only**; implicit vertical viscosity **after** with drag included into implicit solver **(i.e., the drag term is recomputed relative to what BM got before $\Rightarrow$ splitting error)**

Bottom: Bottom drag is computed as a part of implicit vertical viscosity step **before** and for **both** 3D and BM forcing

In all cases BM has bottom drag term which captures its tendency in fast time

$$\partial_t \overline{\mathbf{U}} = \dots \Big[\underbrace{\underbrace{-r_D \cdot \mathbf{u}_{\text{bottom}}}_{\text{drag from 3D mode}} + r_D \cdot \overline{\mathbf{u}}^{m=0}}_{\text{3D}\rightarrow\text{BM forcing}} \Big] - r_D \cdot \overline{\mathbf{u}}$$

so when $\mathbf{u}_{\text{bottom}}$ is updated/corrected by BM, so does the $-r_D \cdot \mathbf{u}_{\text{bottom}}$ term computed from it; above $\overline{\mathbf{U}} = (h + \zeta)\overline{\mathbf{u}}$



Key components for small splitting error:

- No-slip B.C. and bottom drag term **must be included** into implicit vertical viscous solver
- *Both* the total bottom stress term and bottom drag coefficient must be available to BM (i.e. must be precomputed before BM)
- As BM advances vertical integrals of $\overline{u}, \overline{v}$, it should also take into account the incremental changes to bottom drag term, so after u, v^{n+1} are are adjusted by BM, the resultant adjusted bottom drag term is still in balance with the remaining terms as it was before BM run

$$\partial_t \overline{\mathbf{U}} = \ldots \overbrace{\left[\underbrace{-r_D \cdot \mathbf{u}_{\text{bottom}}}_{\text{drag from 3D mode}} + r_D \cdot \overline{\mathbf{u}}^{m=0} \right]}^{\text{3D}\rightarrow\text{BM forcing, "slow"}} \underbrace{-r_D \cdot \overline{\mathbf{u}}}_{\text{"fast"}}$$

Note: $-r_D \cdot \overline{\mathbf{u}}$ **cannot** simulate bottom drag by itself. *The sole purpose* of its presence is to make BM "feel" the incremental change in $-r_D \cdot \delta \overline{\mathbf{u}} = -r_D \cdot \left(\overline{\mathbf{u}} - \overline{\mathbf{u}}^{m=0} \right)$, so the subsequent correction of 3D u, v's by BM (which unavoidably changes the bottom drag if recomputed from the *updated* u, v's) nevertheless is able to predict the change in $-r_D \cdot \mathbf{u}_{\text{bottom}}$.

Flowchart of POM code: `pom2k.f`

Mellor, 2004 POM User's Manual

bottom friction coefficient

$$\mathtt{cbc}_{i,j} = \kappa^2 \Big/ \left[\ln \left(\frac{\Delta z_{i,j,k_b}}{z_{0b}} \right) \right]^2$$

restricted to $\quad 0.0025 < \mathtt{cbcmin} < \mathtt{cbc}_{i,j} < \mathtt{cbcmax} = 1.0$

`profu,v` compute quadratic drag coefficients,

$$\mathtt{tps}_{i+1/2,j} = \overline{\mathtt{cbc}}^x_{i+1/2,j} \sqrt{(u^n)^2 + \left(\left(\overline{\overline{v}}^{x,y} \right)^n \right)^2} \Bigg|_{i+1/2,j,k_b}$$

at u-location $i+1/2, j$ (similarly at $i, j+1/2$) and solve implicit vertical viscosity problem **together with no-slip bottom B.C.**

momentum fluxes at the bottom $\langle wu \rangle$, $\langle wv \rangle$

$$\mathtt{wubot}_{i+1/2,j} = -\mathtt{tps}_{i+1/2,j} \cdot u^{n+1}_{i+1/2,j,k_b}$$

are computed at the very end of **`profu,v`**

$\mathtt{wubot}_{i+1/2,j}$ and $\mathtt{wvbot}_{i,j+1/2}$ are applied to the r.h.s. of BM **during the next** time step, where they are kept **constant** in fast time (no adjustment to bottom drag term within BM)

- built-in delay between bottom drag and BM
- **subject to splitting error** (BM disturbs bottom B.C. after enforcing vertical integrals u, v at next step)
- can be *partially* repaired by introducing adjustment of bottom drag term into BM

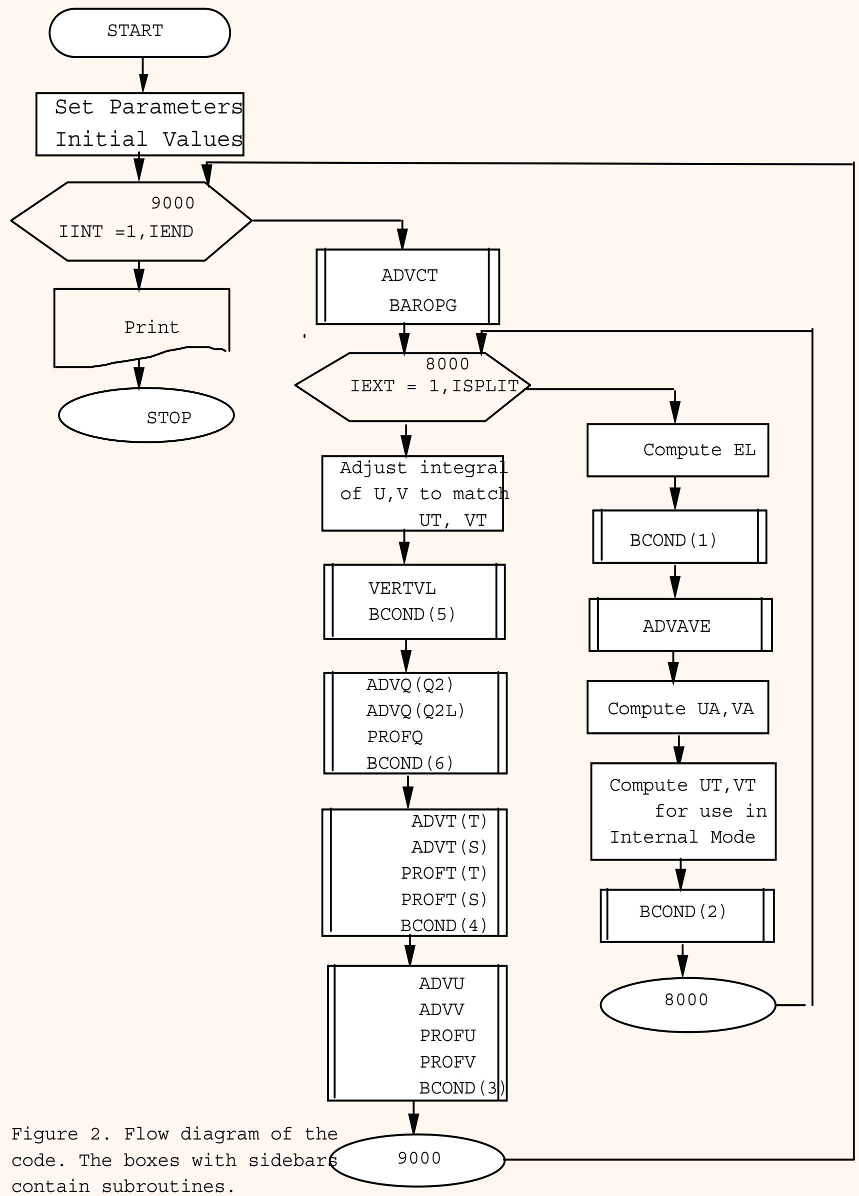


Figure 2. Flow diagram of the code. The boxes with sidebars contain subroutines.

Test problem: Upwelling response

Based on traditional ROMS "Upwelling test" configuration:

- EW periodic channel, 80 km wide;
- $h_{\text{max}} = 25m$ $h_{\text{max}} = 150m$;
- f-plane, $f = -8.26 \times 10^{-5}$ *southern* hemisphere;
- initially flat stratification in T;
- spatially uniform wind, 0.1 N/m^2 stress,
  modulated by $\sin\left(\frac{\pi}{4} \cdot t_{\text{[days]}}\right)$ for $0 < t < 2$ days;
  thereafter constant
- "salinity" is a passive tracer just to illustrate flow, and

either

- analytical vertical viscosity profile, constant in time

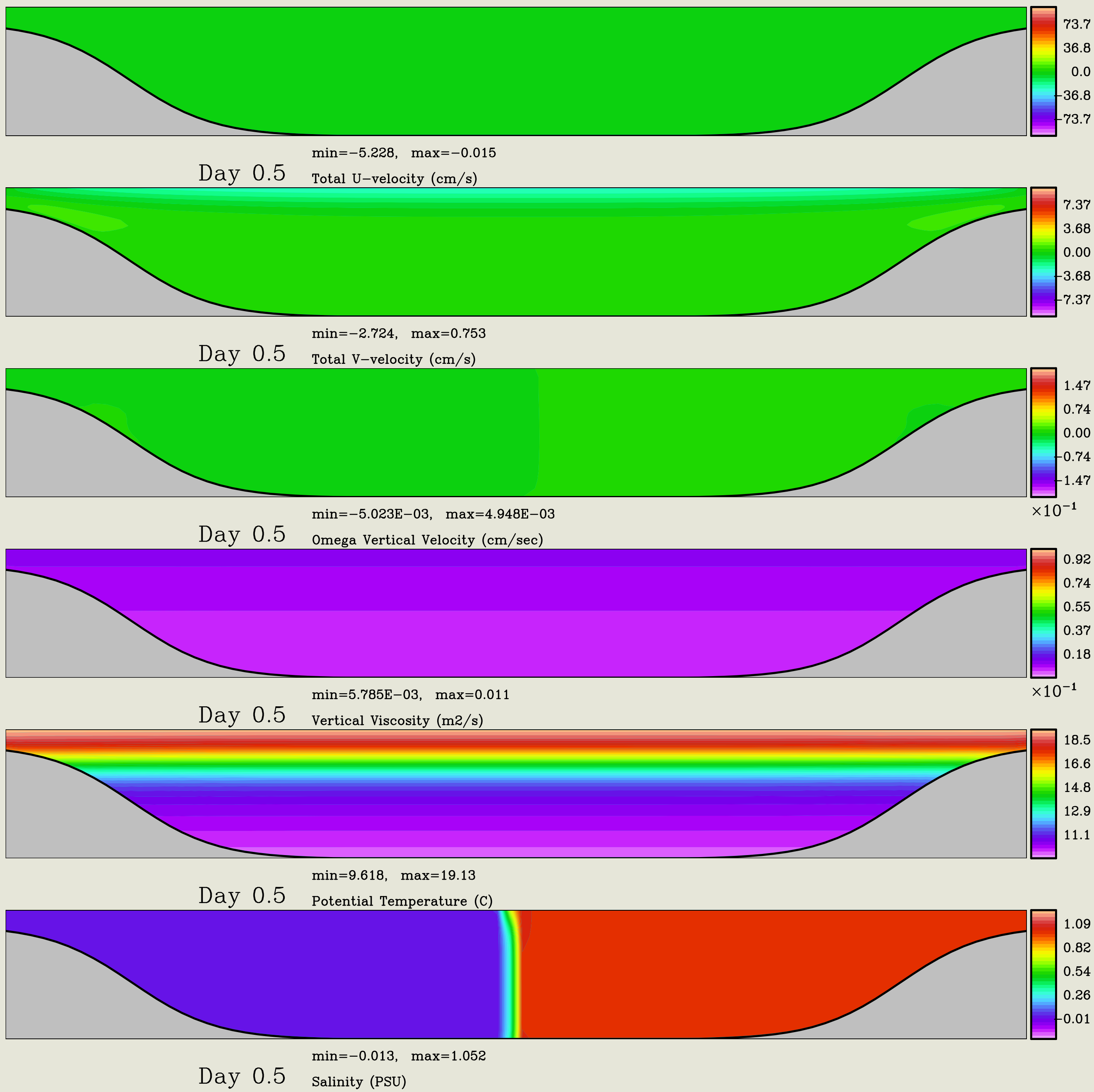
or

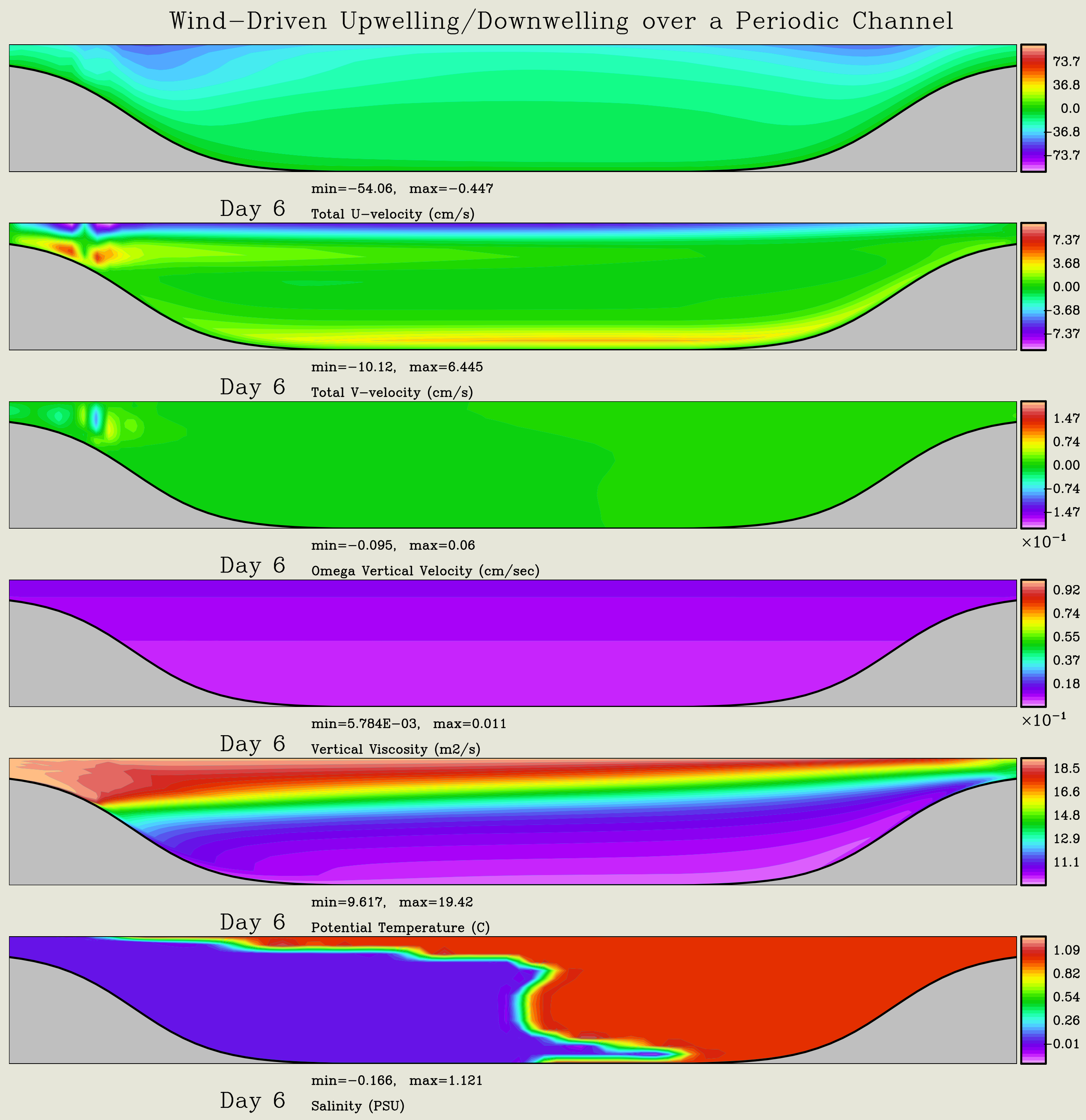
- KPP, both top and bottom, dynamically changing

Wind–Driven Upwelling/Downwelling over a Periodic Channel

Analytical

A_{kv} profile

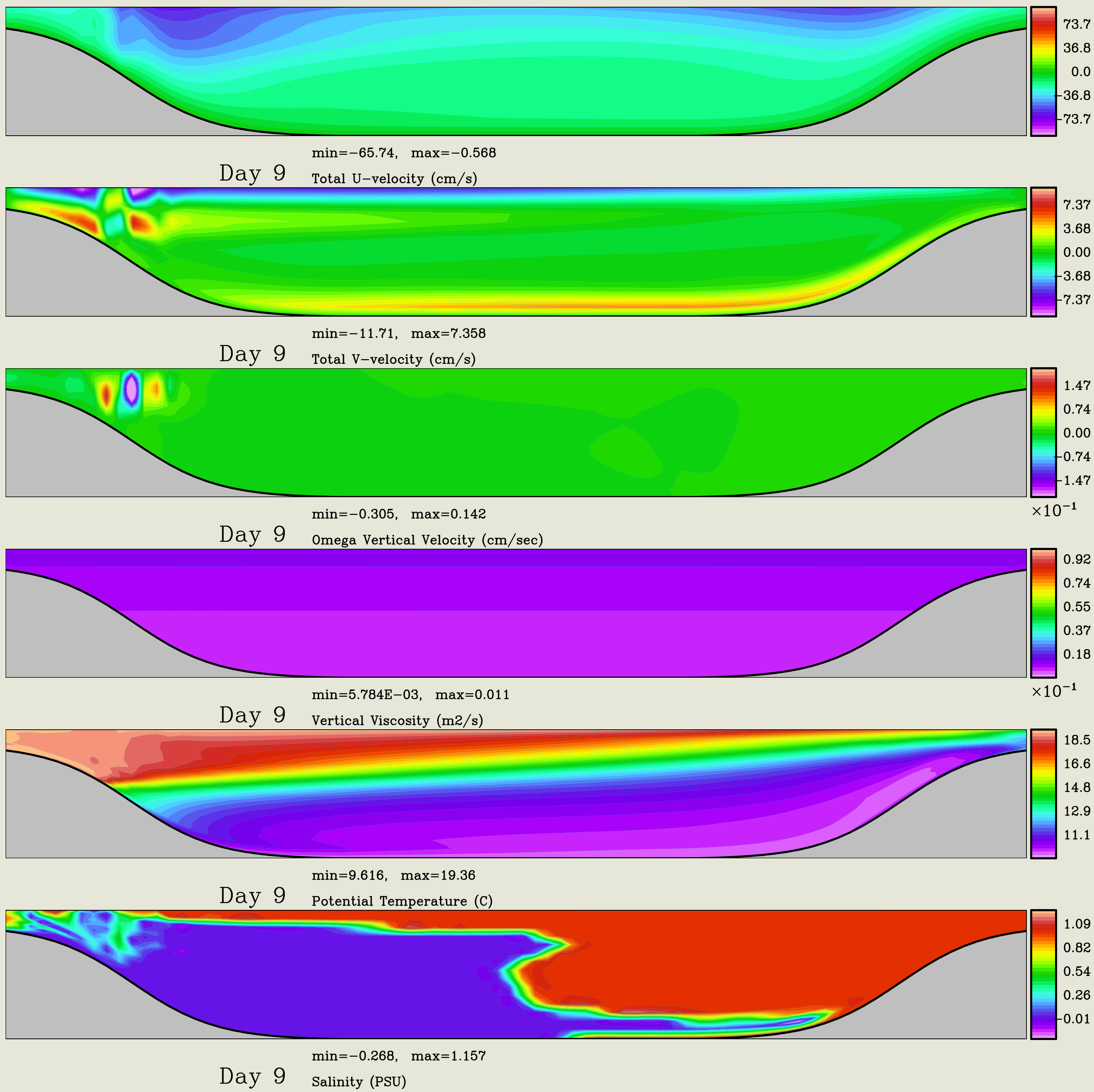




Analytical

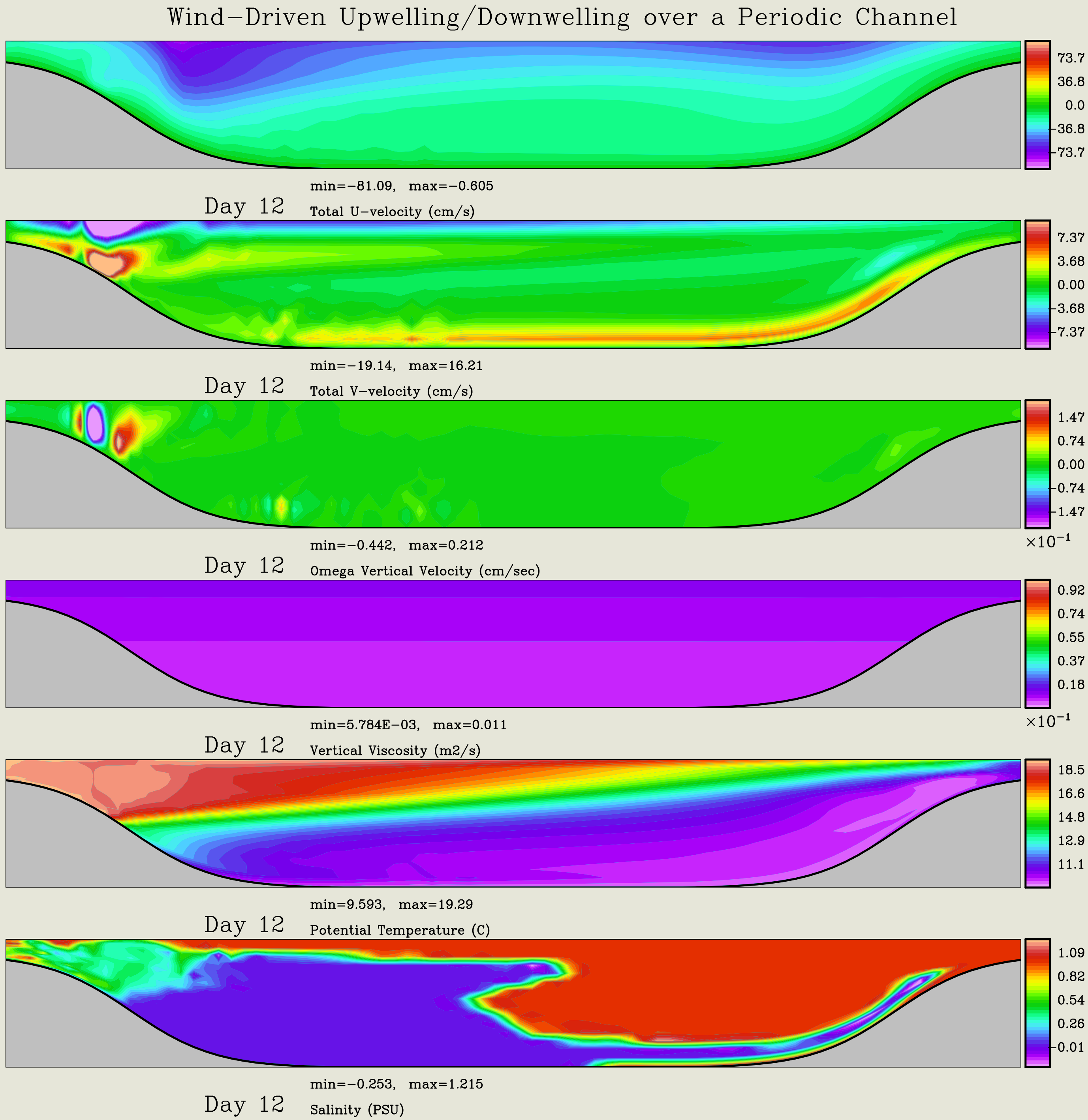
A_{kv} profile

Wind−Driven Upwelling/Downwelling over a Periodic Channel



Analytical

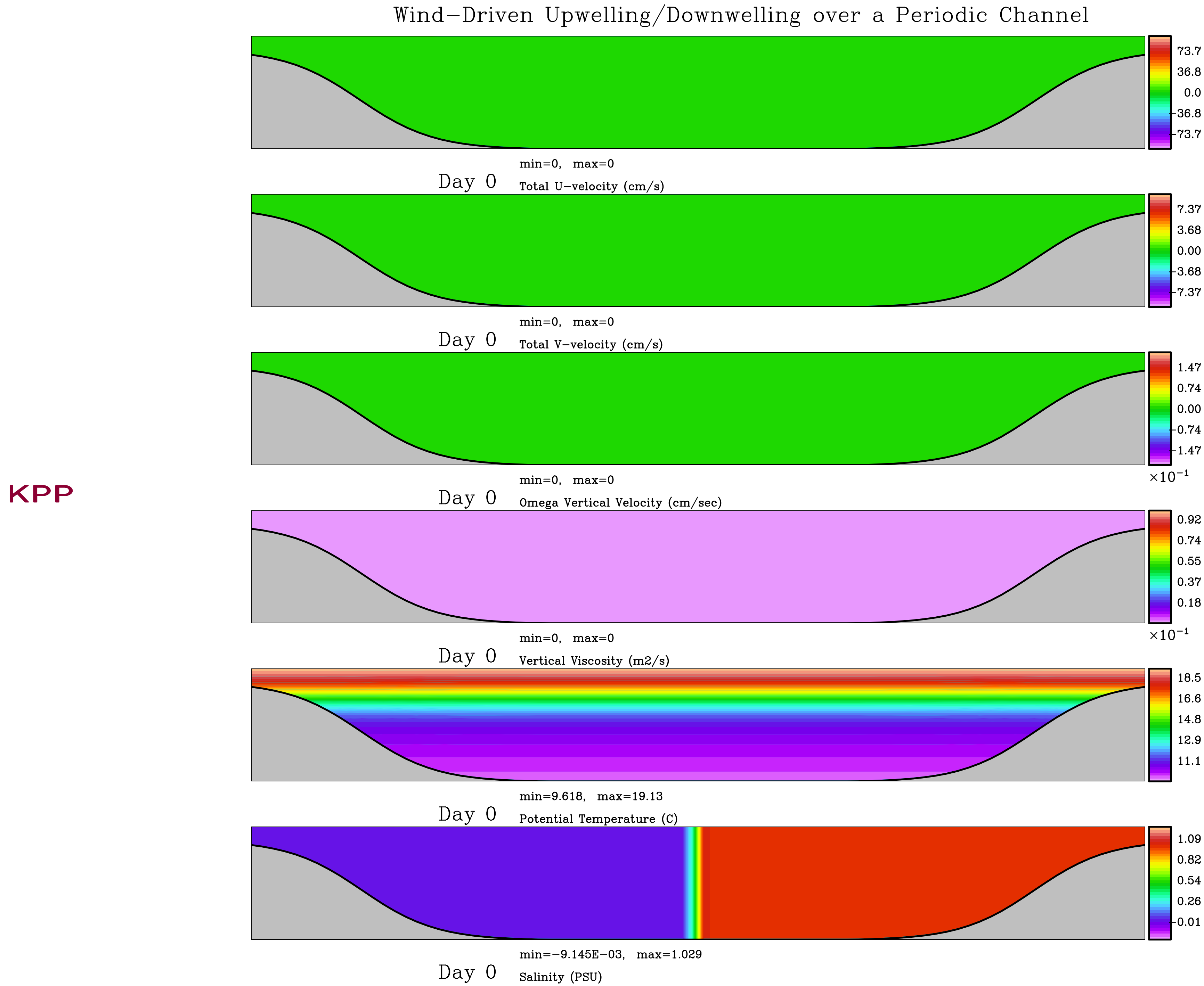
A_{kv} profile



Analytical

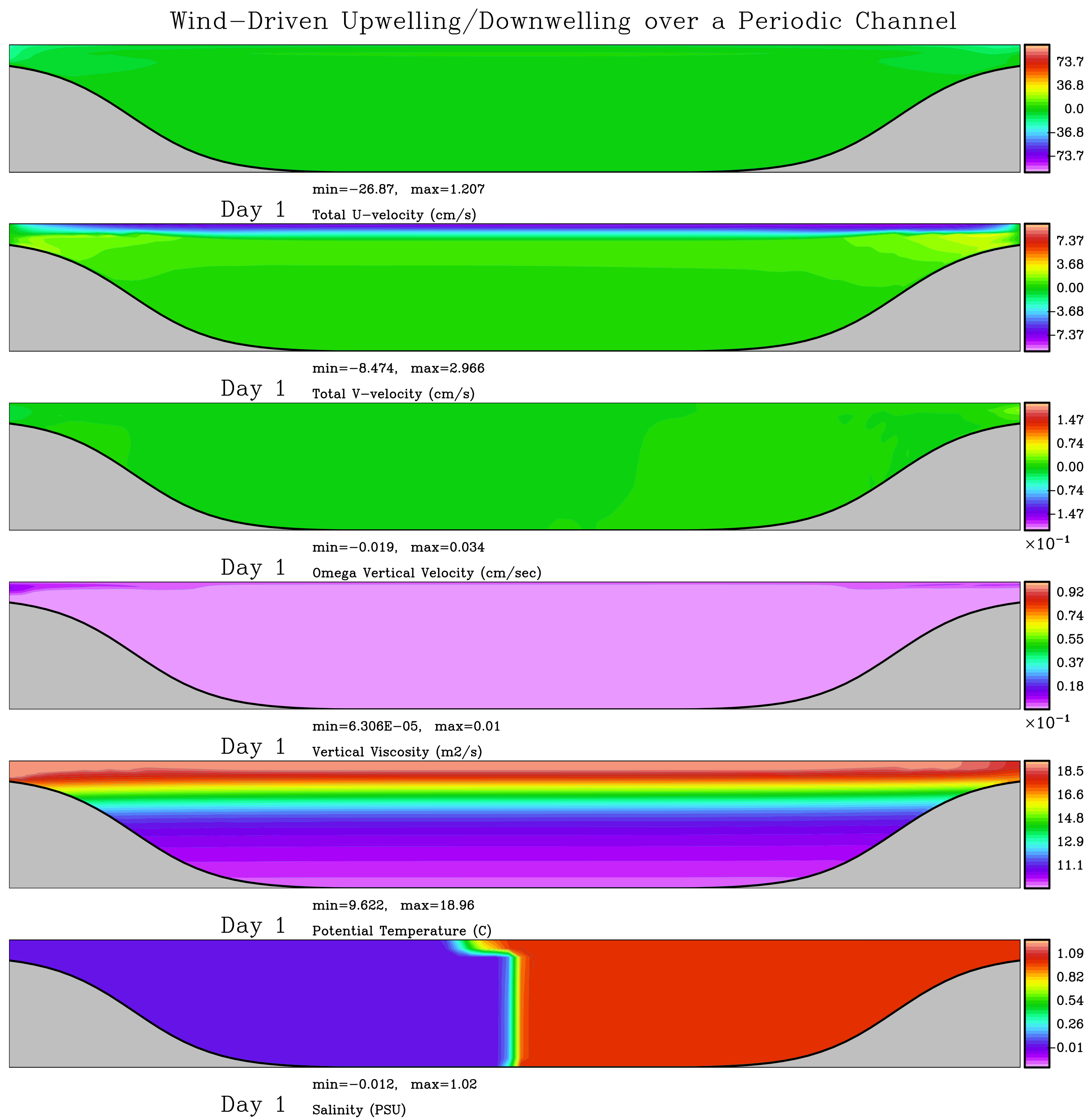
A_{kv} profile

- **same, but with KPP, top and bottom**

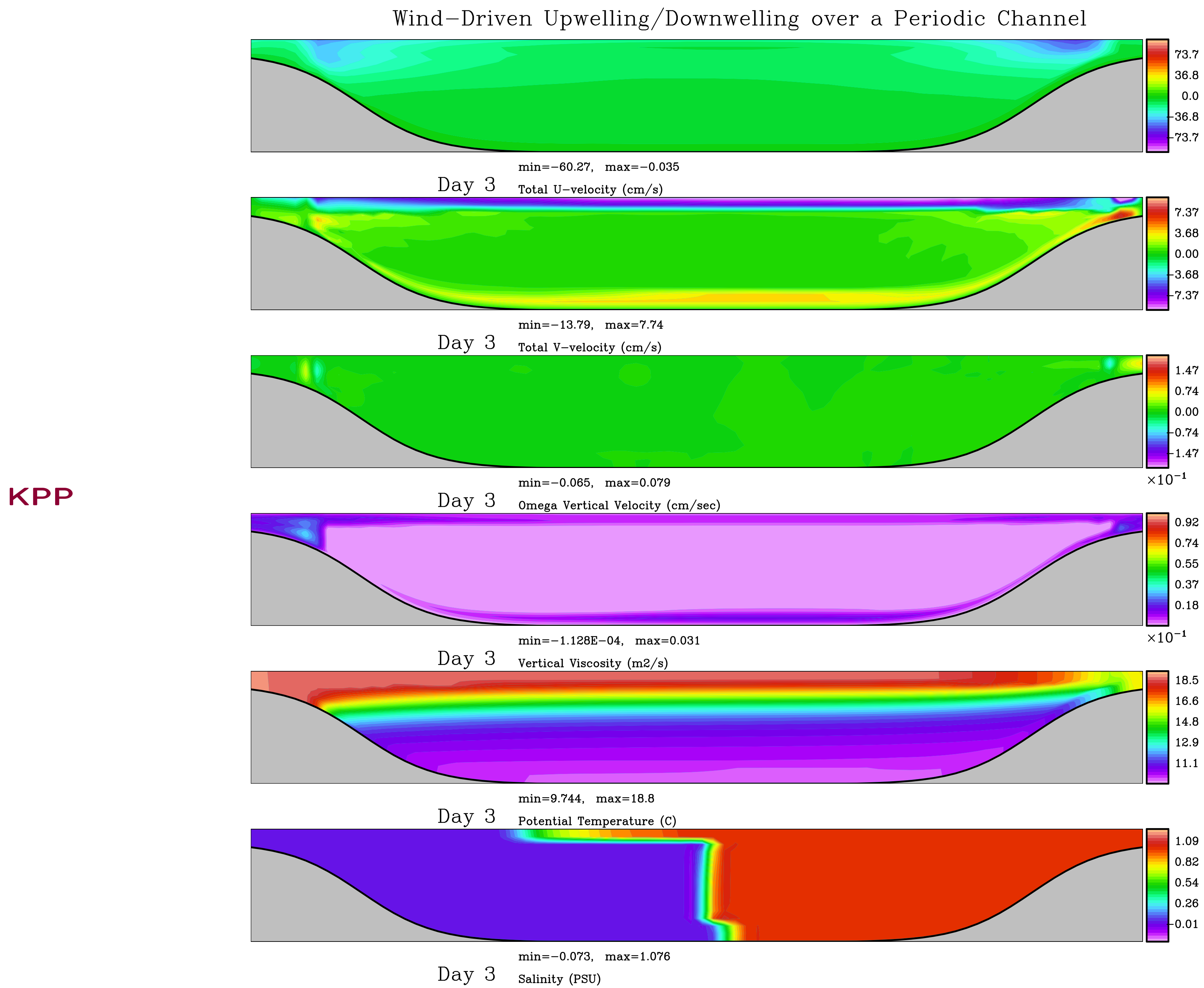


Wind−Driven Upwelling/Downwelling over a Periodic Channel
KPP
73.7
36.8
0.0
-36.8
-73.7
min=0, max=0
Day 0
Total U−velocity (cm/s)
7.37
3.68
0.00
-3.68
-7.37
min=0, max=0
Day 0
Total V−velocity (cm/s)
1.47
0.74
0.00
-0.74
-1.47
×10⁻¹
min=0, max=0
Day 0
Omega Vertical Velocity (cm/sec)
0.92
0.74
0.55
0.37
0.18
×10⁻¹
min=0, max=0
Day 0
Vertical Viscosity (m2/s)
18.5
16.6
14.8
12.9
11.1
min=9.618, max=19.13
Day 0
Potential Temperature (C)
1.09
0.82
0.54
0.26
0.01
min=−9.145E−03, max=1.029
Day 0
Salinity (PSU)

KPP

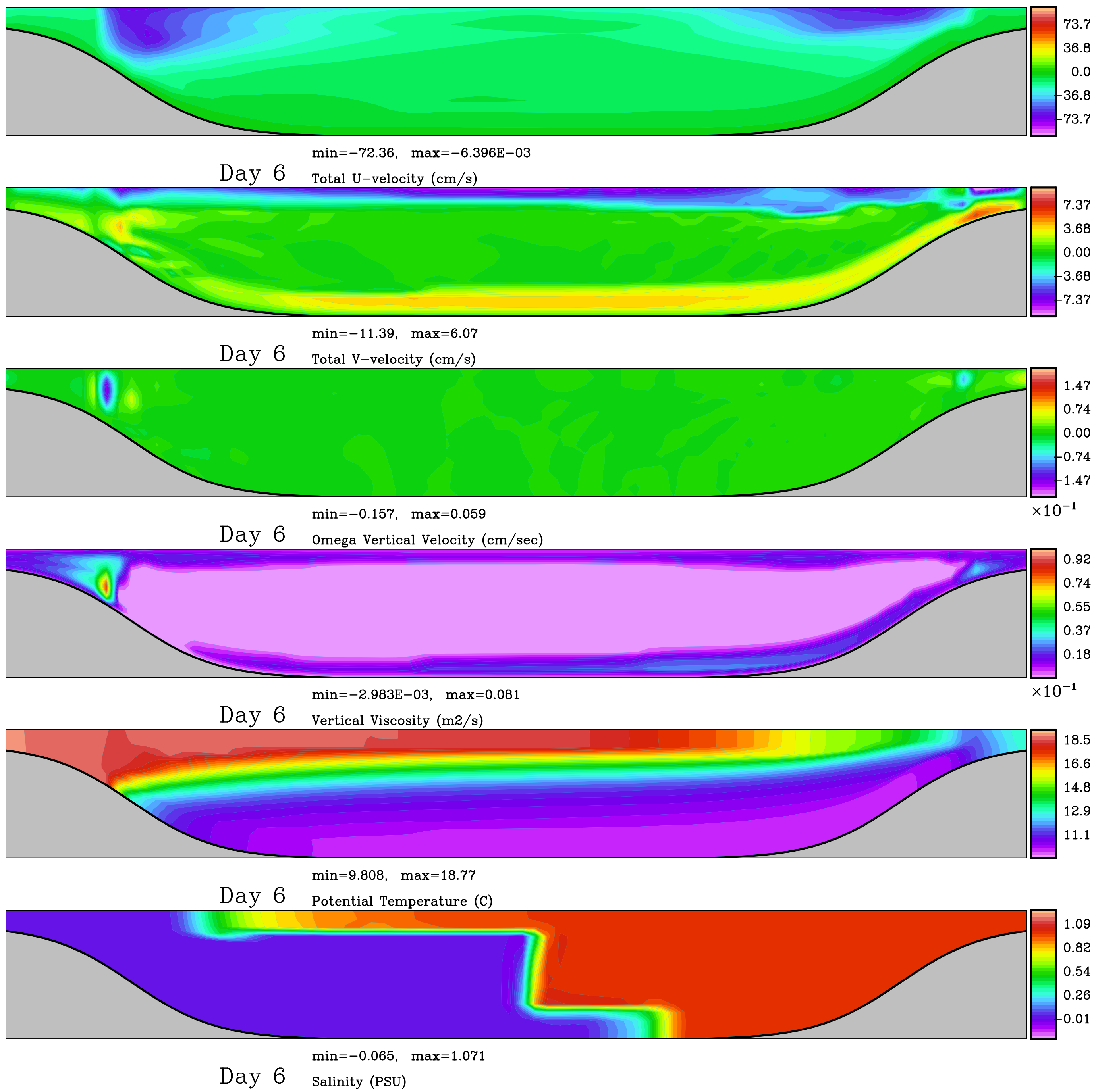


Wind-Driven Upwelling/Downwelling over a Periodic Channel
73.7
36.8
0.0
-36.8
-73.7
min=-26.87, max=1.207
Day 1
Total U-velocity (cm/s)
7.37
3.68
0.00
-3.68
-7.37
min=-8.474, max=2.966
Day 1
Total V-velocity (cm/s)
1.47
0.74
0.00
-0.74
-1.47
×10⁻¹
min=-0.019, max=0.034
Day 1
Omega Vertical Velocity (cm/sec)
0.92
0.74
0.55
0.37
0.18
×10⁻¹
min=6.306E-05, max=0.01
Day 1
Vertical Viscosity (m2/s)
18.5
16.6
14.8
12.9
11.1
min=9.622, max=18.96
Day 1
Potential Temperature (C)
1.09
0.82
0.54
0.26
0.01
min=-0.012, max=1.02
Day 1
Salinity (PSU)


Wind–Driven Upwelling/Downwelling over a Periodic Channel
73.7
36.8
0.0
-36.8
-73.7
min=−60.27, max=−0.035
Day 3
Total U–velocity (cm/s)
7.37
3.68
0.00
-3.68
-7.37
min=−13.79, max=7.74
Day 3
Total V–velocity (cm/s)
1.47
0.74
0.00
-0.74
-1.47
$\times10^{-1}$
min=−0.065, max=0.079
Day 3
Omega Vertical Velocity (cm/sec)
KPP
0.92
0.74
0.55
0.37
0.18
$\times10^{-1}$
min=−1.128E−04, max=0.031
Day 3
Vertical Viscosity (m2/s)
18.5
16.6
14.8
12.9
11.1
min=9.744, max=18.8
Day 3
Potential Temperature (C)
1.09
0.82
0.54
0.26
-0.01
min=−0.073, max=1.076
Day 3
Salinity (PSU)

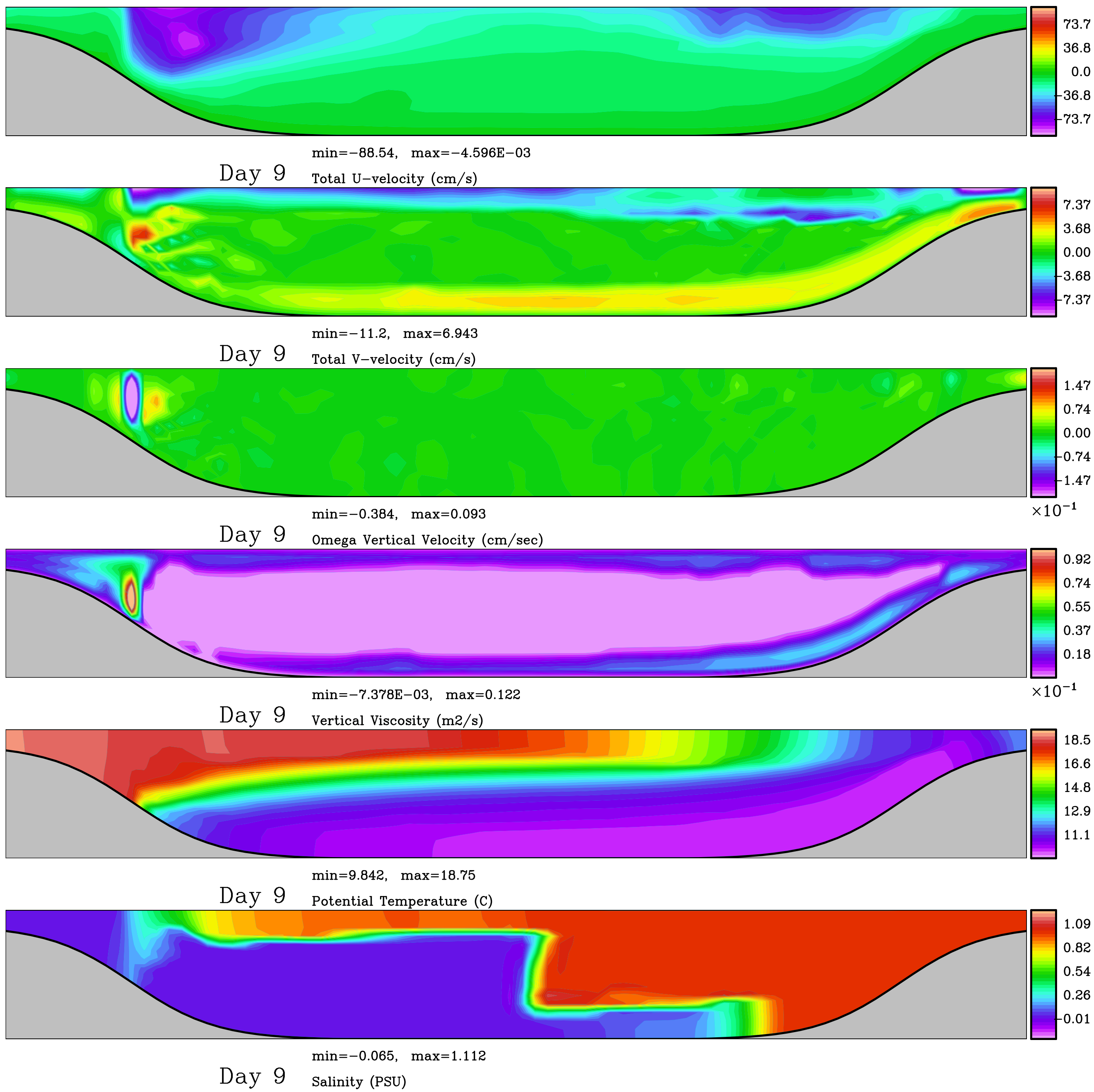
Wind−Driven Upwelling/Downwelling over a Periodic Channel

KPP



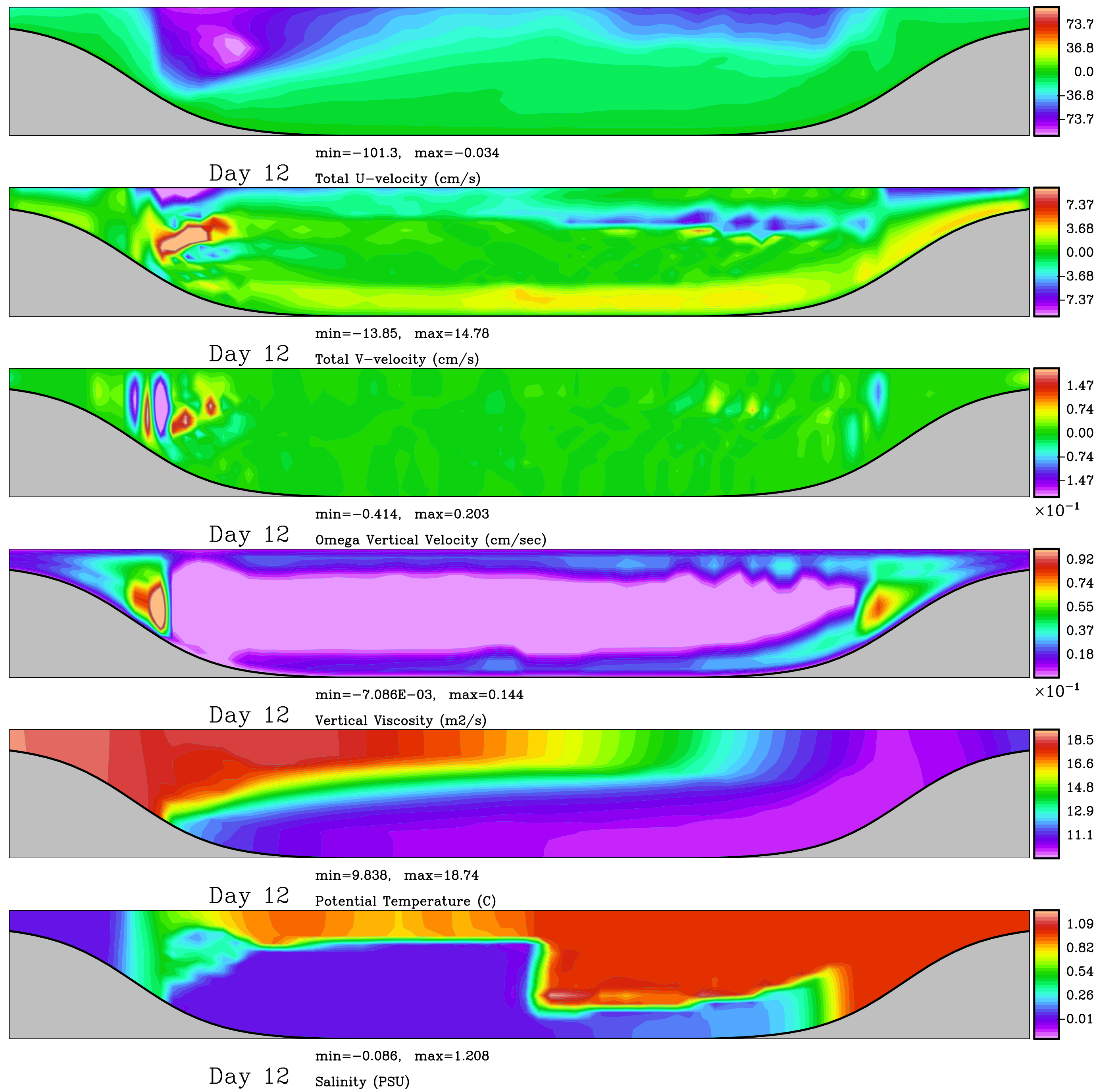
Wind−Driven Upwelling/Downwelling over a Periodic Channel

KPP

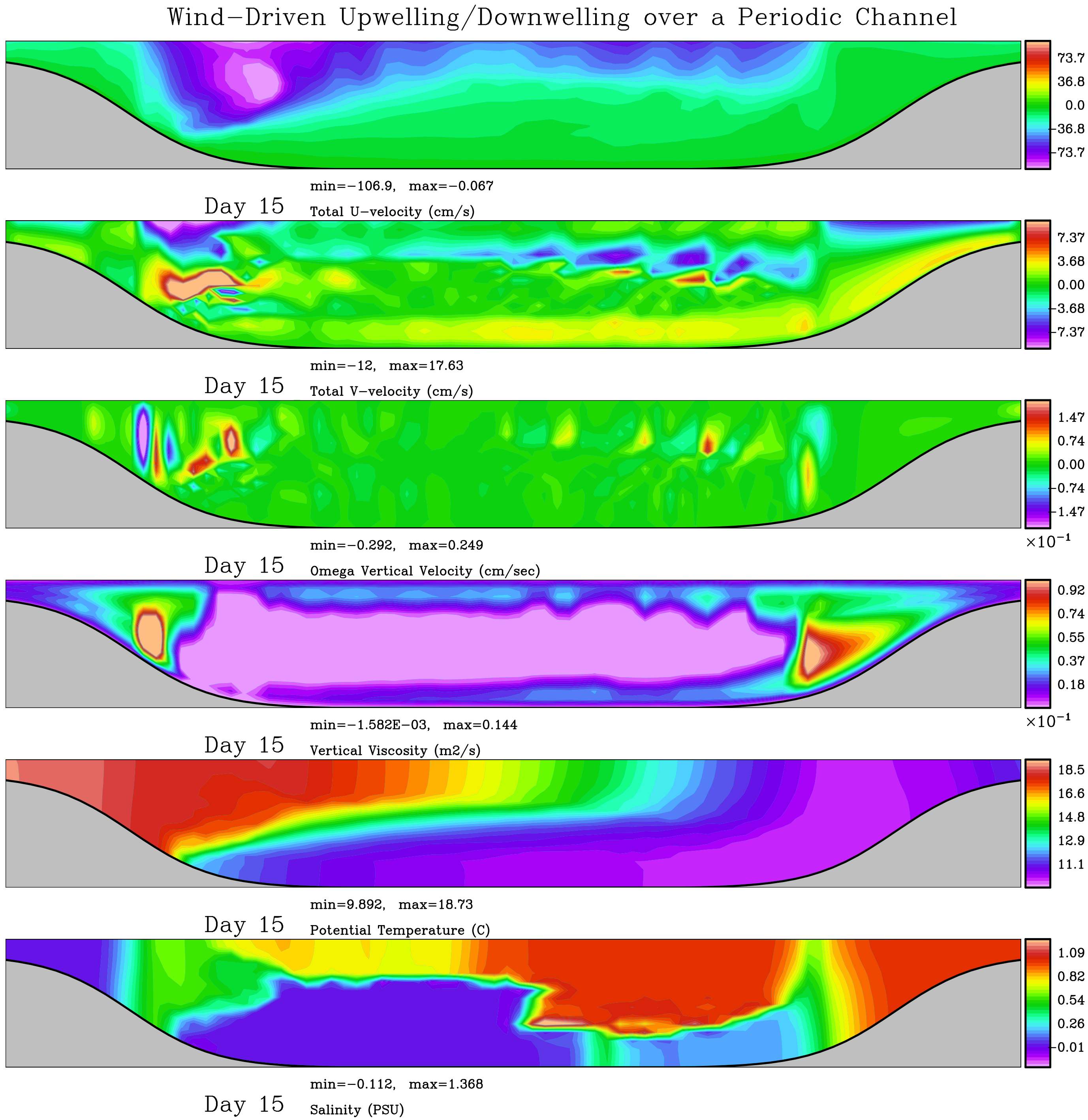


Wind−Driven Upwelling/Downwelling over a Periodic Channel

KPP



KPP



Wind−Driven Upwelling/Downwelling over a Periodic Channel
73.7
36.8
0.0
-36.8
-73.7
min=−106.9, max=−0.067
Day 15
Total U−velocity (cm/s)
7.37
3.68
0.00
-3.68
-7.37
min=−12, max=17.63
Day 15
Total V−velocity (cm/s)
1.47
0.74
0.00
-0.74
-1.47
×10⁻¹
min=−0.292, max=0.249
Day 15
Omega Vertical Velocity (cm/sec)
0.92
0.74
0.55
0.37
0.18
×10⁻¹
min=−1.582E−03, max=0.144
Day 15
Vertical Viscosity (m2/s)
18.5
16.6
14.8
12.9
11.1
min=9.892, max=18.73
Day 15
Potential Temperature (C)
1.09
0.82
0.54
0.26
0.01
min=−0.112, max=1.368
Day 15
Salinity (PSU)

Realistic example: USWC L4 Palos Verdes configuration

USWC L4 Palos Verdes grid configuration

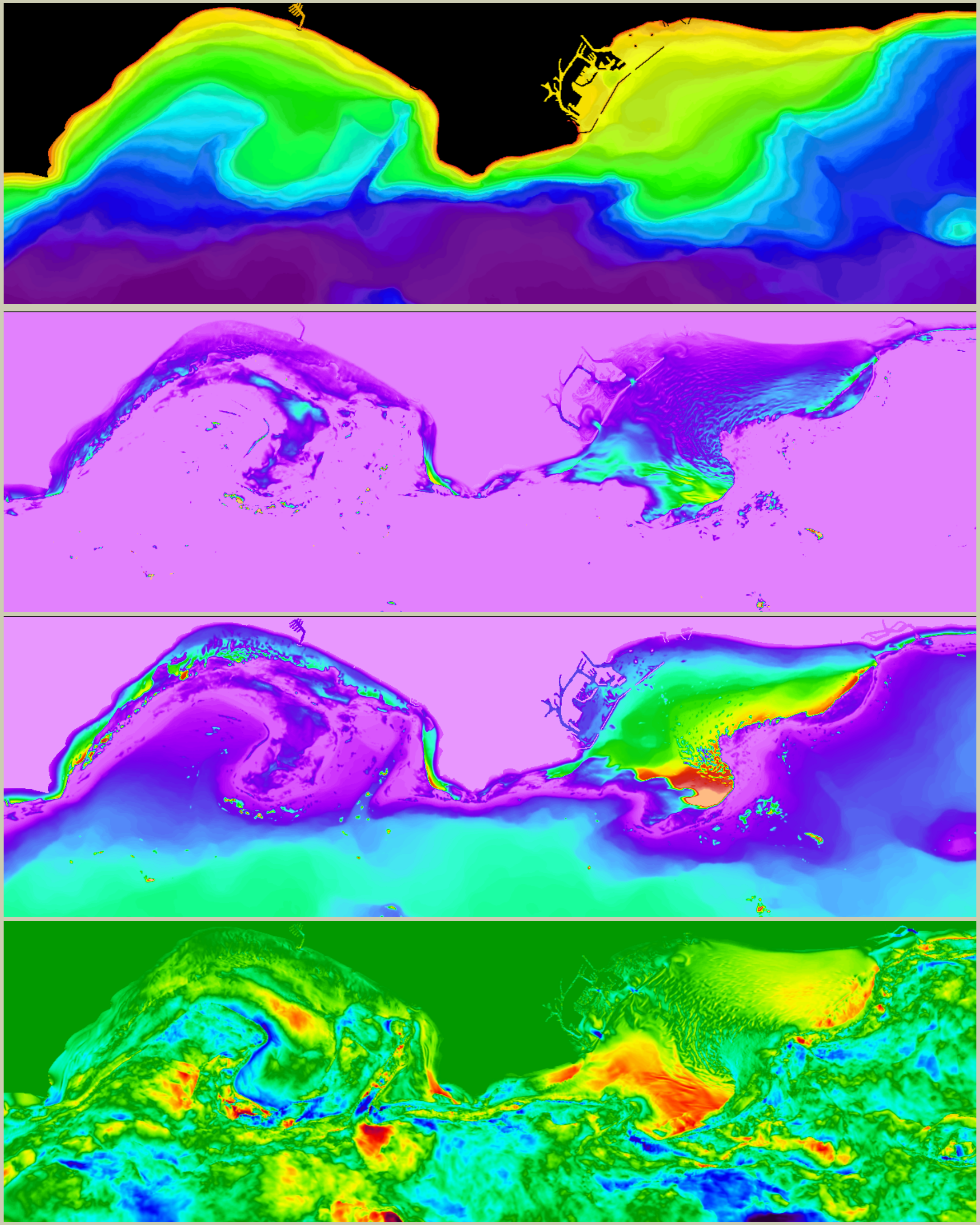
$\Delta x = 75m$

h, logscale, $\mathtt{min} = 1.5m$ $\mathtt{max} = 900m$

r_D, max=0.01

h_{bbl} by KPP max=$50m$

u, $\pm 0.2\, m/s$



Summary:

- Overall classical operator splitting dilemma

$$\partial_t \mathbf{u} = \mathcal{R}(\mathbf{u}) \qquad \text{where} \qquad \mathcal{R}(\mathbf{u}) = \mathcal{R}_1(\mathbf{u}) + \mathcal{R}_2(\mathbf{u}) \qquad \text{both are stiff, but}$$

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t \cdot \mathcal{R}\left(\mathbf{u}^{n:n+1}\right)$$

is not practical because of complexity (implicitness), so instead

$$\mathbf{u}' = \mathbf{u}^n + \Delta t \cdot \mathcal{R}_1(\mathbf{u}^{n:\prime}) \qquad \text{followed by} \qquad \mathbf{u}^{n+1} = \mathbf{u}' + \Delta t \cdot \mathcal{R}_2(\mathbf{u}^{\prime:n+1})$$

$$\mathbf{u}^{n+1} = [1 + \Delta t \cdot \mathcal{R}_2(.)] \cdot [1 + \Delta t \cdot \mathcal{R}_1(.)]\, \mathbf{u}^n$$

$$[1 + \Delta t \cdot \mathcal{R}_2(.)] \cdot [1 + \Delta t \cdot \mathcal{R}_1(.)] \neq [1 + \Delta t \cdot \mathcal{R}_1(.)] \cdot [1 + \Delta t \cdot \mathcal{R}_2(.)]$$

resulting in $\mathcal{O}(\Delta t)$ operator splitting error.
Especially inaccurate in near cancellation $R_1 \approx -R_2$ situation (balance).

- reminiscent of implicit no-slip boundaries + pressure-Poisson projection method for incompressible flows
- Requires substantial redesign of ROMS kernel
- somewhat encourages **anti-modular** code design
- Possible only in corrector-coupled and Generalized FB variants of ROMS kernels
- Incompatible (or at least hard to implement) in Rutgers ROMS because of forward extrapolation of r.h.s. terms for 3D momenta (AB3 stepping) and extrapolation of 3D→BM forcing terms which is not compatible with having stiff terms there
- Incompatible with predictor-coupled variant of ROMS kernel (currently used by AGRIF), because of extrapolation of 3D→BM forcing, and having BM too early the computing sequence (implicit vertical viscosity step is done only after predictor step for tracers which is after BM)
- **Must have, long overdue**