1 The Island Wake Problem

The purpose of this note is to provide a technical explanation for posing the Island Wake Problem. The particular formulation below generally parallels the formulation of Charles Dong reported in Venice, but may not coincide with it in the details of setting up density and velocity profiles in the initial state and inflow conditions, as well as in the choice of outflow of boundary condition on the east side.

All parts of code specifically related to this setup are activated by CPP-switch `ISWAKE`.

1.1 Initial Conditions

Velocity, density and free-surface field for initial conditions and west-side (inflow) boundary are specified to respect geostrophic balance (via thermal-wind balance equation), which is also a stationary solution to primitive equations in Cartesian coordinates

$$\frac{\partial u}{\partial t} + ... - fv = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} \qquad \frac{\partial v}{\partial t} + ... + fu = -\frac{1}{\rho_0}\frac{\partial P}{\partial y}\,. \tag{1}$$

The geostrophic balance in y-direction reads

$$+fu = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} = -\frac{1}{\rho_0}\frac{\partial}{\partial y}\int_z^\zeta g\rho\,dz'\,, \tag{2}$$

which leads to the thermal-wind balance equation

$$f\frac{\partial u}{\partial z} = \frac{g}{\rho_0}\cdot\frac{\partial \rho}{\partial y}\,. \tag{3}$$

Now suppose u and ρ are set via the same function $\Phi(\xi)$,

$$u = U_0\Phi(\xi) \qquad \rho = R_0\Phi(\xi) \tag{4}$$

where U_0 and R_0 are amplitudes of velocity and density perturbations, while $\xi = z + Ay$ is a "sloped" cross-isopycnic coordinate (isosurfaces of $\xi = const$ are isopycnals). Then to satisfy (3) one needs to satisfy

$$fU_0\Phi'(\xi)\cdot\frac{\partial \xi}{\partial z} = \frac{g}{\rho_0}R_0\Phi'(\xi)A\,, \tag{5}$$

where $\Phi'(\xi)$ denotes natural derivative of $\Phi(\xi)$ with respect to its argument. On an f-plane the isopycnal slope is determined by

$$A = fU_0\frac{\rho_0}{gR_0}\,. \tag{6}$$

Practical setting

$$\Phi(z,y) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{z - z_0 + Ay}{\sigma}\right) \tag{7}$$

where z_0 is thermocline depth, and σ is thermocline thickness.

Free-surface elevation is to cancel out pressure gradient force at large depth to keep deep fluid at rest. Since

$$\int \tanh(x)\, dx = \ln\Big(\cosh(x)\Big) \tag{8}$$

one can derive

$$\int_{-\infty}^{0} \left\{\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{z-z_0}{\sigma}\right)\right\} dz = \frac{1}{2}\sigma \,\ln\left[\cosh\left\{\frac{z-z_0}{\sigma}\right\}\cdot\exp\left\{\frac{z}{\sigma}\right\}\right]\Bigg|_{-\infty}^{0} = \frac{1}{2}\sigma\ln\left[1+\exp\left\{-\frac{2z_0}{\sigma}\right\}\right].$$

where it can be easily verified that limit of step function, $\sigma \to 0$, the above integral goes to its limit of $-z_0$ as it should be. From above we specify initial free-surface field

$$\zeta(y) = \frac{R_0}{2\rho_0}\sigma\ln\left[\frac{1+\exp\left\{2\left(Ay-z_0\right)/\sigma\right\}}{1+\exp\left\{-2z_0/\sigma\right\}}\right] \tag{9}$$

which makes total (barotropic + baroclinic) pressure gradient vanish at infinite depth.

Initial conditions (4) with (7) and (9), as well as $v = 0$ is specified everywhere as it would be no island within the computational domain. After which land mask is applied to all these fields. Consequently, normal velocity component exhibits sudden change in the vicinity of the land resulting in rather violent initial adjustment with intense generation of internal and surface gravity waves. These, however quickly leave the computation domain (radiated out) resulting in settling of current around the island in a period of one to few days.

Practical setting of parameters $z_0 = -120m$ and $\sigma = 60m$ $U_0 = 0.2m/sec$, and $R_0 = 3kg/m^3$, which approximately corresponds to temperature difference across the thermocline of $10deg\ C$. The phase speed of the internal wave of the baroclinic mode (as measured by observing the initial adjustment process) is about 1.6 m/sec, which corresponds to the deformation radius $r_D = C/f = 16km$, which is comparable to the size of the island.

1.2 Bottom Topography

Either cylindrical island with vertical side walls, or an island with Gaussian shaped topography are available in this setup. The cylinder setup has constant depth, $h = \max$, and land mask set to "land" inside the circular area $r \le r_i$. The Gaussian topography is set by

$$h(r) = \begin{cases} h_{\min}\,, & r \le r_i \\ h_{\max} - (h_{\max}-h_{\min})\exp\left\{-(r-r_i)^2/4r_i^2\right\}\,, & r > r_i \end{cases} \tag{10}$$

This code is activated by CPP-switch `GAUSSIAN_SLOPE_ISLAND` defined locally inside file `ana_grid.F`. It should be noted that generated this way topography h has zero topographic slope at the coastline of the island, which is numerically desirable for accurate pressure gradient computation in the proximity of land mask. The radius of above-surface part of the island r_i is set to either 10 or 5 km for cylinder and Gaussian cases in the computed examples. Increasing it beyond 10 and 7.5 km is undesirable because wake starts interfering with the side walls. Note that in our setup the width of computational domain is 80 km; $h_{\max} = 500m$, and $h_{\min} = 20m$; vertical S-coordinate stretching parameter is chosen to 2.5, resulting in approximately the same number of vertical grid levels above and below median thermocline depth of 120 m.

1.3 Side Boundary Conditions

The side boundary data is provided by saving one row on boundary points on the western and eastern side from the initial state (4), (7) and (9) and keeping them constant during the whole simulation. They are stored in the code in special small-sized arrays, thus eliminating the need for 3D "climatological" arrays, as often done in many applications of ROMS. Side boundary data for v component is zero field.

The north and south boundaries are free-slip walls with zero-normal gradient condition for temperature field. They require no input of external data.

The western boundary is formulated as an inflow boundary with prescribed density field and velocity components (v=0).

The inflow boundary condition for the barotropic component is converted into "constant flux" rather than "constant velocity" via

$$\overline{u} = \overline{u}_* \cdot \frac{h+\zeta_*}{h+\zeta} \approx \overline{u}_* \left(1 + \frac{\zeta_* - \zeta}{h}\right) \tag{11}$$

where $\overline{u}$ and ζ from from model solution, h is unperturbed depth of water column, while $\overline{u}_*$ and ζ_* are externally supplied data. As the result, $\overline{u}$ at the western boundary is slightly decreased if modeled ζ exceeds ζ_* and vice-versa, but the net barotropic mass flux is kept constant and independent from zeta.

At the eastern boundary radiation (Orlanski) conditions are used. The *free*-radiation (no feeding back of external data) condition is applied to temperature fields, while gentle restoring [$\tau = 1$ day] is applied for velocity both components. Given the characteristic time scales for this problem, this value is considered a mild. Without this feedback the model exhibits a weak instability, slowly developing unphysical flows along either northern (typically) or southern boundaries.

In addition to that, the outgoing barotropic velocity component normal to the boundary is subjected to adjustment to compensate mismatch between the modeled and external free-surface elevation. This is done by applying "reduced" pressure-gradient-like term

$$\left.\frac{\partial \overline{u}}{\partial t}\right|_{\text{east}} = \dots - \frac{1}{100} \cdot g \frac{\zeta_* - \zeta}{\Delta x} \tag{12}$$

to the boundary value $\overline{u}$ after the completion the standard radiation algorithm. The presence of this term tends to minimize the mismatch between the modeled ζ and the externally supplied ζ_* via the natural feedback throughout barotropic momentum and continuity equation. The reduction factor (1/100 above) is empirical, and this practical setting is sufficient to avoid keep the deviation of mean free-surface elevation within the millimeter range, while having no observable interference with the radiation boundary algorithm. Code runs normally with setting of this value up to 1, however doing so results in strong reflection of barotropic gravity waves from the eastern boundary.

Standard hard constraint for volume conservation (available under CPP-switch OBC_VOL_CONS is not used in this setup. Practical experience shows that using it results in rather crude interference with open boundary algorithm.

Neumann boundary conditions are applied to the free surface elevation at all boundaries, hence the only use of externally supplied free-surface value ζ_* is to adjust $\overline{u}$ at the western and eastern boundaries as specified above.

There are no sponges or nudging layers long the boundaries of any kind.

The 3D code runs without any explicit viscosity and does not require it for numerical reasons, relying exclusively on the third-order upstream-biased scheme. Note that I typically keep CPP switch UV_VIS2 **undefined** while working on the code to make things go a little bit faster. It should be defined in cppdefs.h if explicit viscosity is desired, e.g., studying Reynolds number sensitivity.

The Island Wake problem is also available in 2D setup, however it is of very limited use because only second-order centered advection scheme is available in UCLA ROMS in purely 2D setup. Consequently, for this problem Reynolds number cannot exceed 5 $\times$ number of grid points across the island, which is practically very low to study any turbulent regime.